The Insurance is the Lemon: Failing to Index Contracts

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ABSTRACT

We model the widespread failure of contracts to share risk using available indices. A borrower and lender can share risk by conditioning repayments on an index. The lender has private information about the ability of this index to measure the true state that the borrower would like to hedge. The lender is risk averse and thus requires a premium to insure the borrower. The borrower, however, might be paying something for nothing if the index is a poor measure of the true state. We provide sufficient conditions for this effect to cause the borrower to choose a nonindexed contract instead.

Keywords: risk sharing, adverse selection, indexation, insurance.

JEL Codes: D86, G22, G32.

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A central implication of the literature on financial contracting is that agents should structure contracts to share risk as efficiently as possible. In many financial markets, standard contracts are simple and do not include risk-sharing arrangements that condition payments on publicly available indices. A leading example of this phenomenon is the mortgage market. In this market, homeowners are exposed to the risk that their homes will decline in value. Lenders are arguably better equipped to bear this risk and could insulate homeowners against declines in house prices by making mortgage repayment terms contingent on a house-price index. These types of mortgage contracts have been widely proposed as a solution to problems facing the mortgage market, such as the subprime default crises of 2007, but have failed to supplant the standard mortgage. Two common explanations for this type of market failure are that the space of feasible contracts is incomplete (Hart and Moore (1988)) or the transaction costs associated with implementing risk-sharing contracts entails are high transaction. Neither of these explanations applies, however, when indices are available that would allow agents to share risk efficiently and that are almost costless to contract upon.

In this paper we develop a model in which the failure to condition on indices and thus efficiently share risk is an equilibrium outcome resulting from asymmetric information. In our model, an agent, who we call the borrower, seeks financing from a set of lenders. This financial contract must be written in view of potential conflicts of interest between the lender and the borrower, which are related to an “internal,” or idiosyncratic, state. For example, this internal state could represent the hidden ability of a mortgage borrower to make payments to the lender. At the same time, there may be some benefits of risk-sharing between the lender and the borrower over some imperfectly measured state (e.g., local area house prices). We refer to this state as “external” to indicate that it is unaffected by the actions of the lenders and the borrower. The external state is not directly observable. To realize any risk sharing benefits, the contracts must condition on some potentially imperfect measurement of the state, which we call an index (e.g., a house price index). Lenders know the true joint distribution of the index and the external state (i.e., the quality of the index), while the borrower does not. In effect, the borrower faces an adverse selection problem over basis risk when lenders offer an indexed contract.

At least two equilibria can arise in the model. In the first type of equilibrium, which we refer to as the full-information optimal contracts equilibrium, all lenders offer a contract featuring the optimal amount of insurance conditional on the true quality of the index. The full-information optimal contracts equilibrium exists when there is competition between lenders and features no loss in efficiency due to asymmetric information about the index. In the second type of equilibrium, which we refer to as the noncontingent contracts equilibrium, all lenders offer a contract that does not condition on the index. To see why such an equilibrium can arise, consider the borrower’s response when a single lender deviates and offers a contingent contract. To at least break even on such a contract, the lender must charge the borrower an insurance premium. At the same time, the borrower may

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1See, for example, the “Shared Responsibility Mortgage” proposed by Mian and Sufi (2015), in which interest and principal payments are contingent upon local house price indices, or the “Shared-Equity Mortgage” proposed by Caplin, Carr, Pollock, Yi Tong, Tan, and Thampy (2007), in which a borrower receives a second mortgage where payment is due only upon the sale of the house and is contingent on house value.
be concerned that the index is in fact uncorrelated with the risk that she is aiming to insure, that is, that the basis risk for the contract is too high to justify the premium. Lenders that know the basis risk is high are happy to offer insurance and charge a high premium because the insurance is cheap for them to provide (precisely because the basis risk is high). As a result, the borrower will reject the indexed contract in favor of a standard noncontingent contract. We note that the noncontingent contracts equilibrium exists even though the contracting space allows for the use of an index, there are no transactions costs, and lenders make competing offers.

To illustrate the intuition behind these two equilibria, suppose that there are two equally likely external states, “good” and “bad.” Now suppose that a borrower receives offers of one dollar of financing from several competing lenders. The borrower is risk-averse with respect to the external state, meaning that her expected marginal value of a dollar is 1/2 in the good state and 3/2 in the bad state. The lenders are also risk averse, but less so than the borrower. Their expected marginal value of a dollar is 3/4 in the good state and 5/4 in the bad state. Lenders can offer contracts that are contingent upon some index but not upon the true external state directly. The index can be “high quality,” in which case it is perfectly correlated with the true underlying state, or “low quality,” in which case it is independent of the true state and hence unrelated to the either the borrower’s or the lender’s preferences. The borrower believes that these two cases are equally likely but lenders observe the quality of the index before making their offers. Finally, the lenders cannot offer contracts that specify positive transfers from the lender to the borrower.

Suppose that the lenders make the following offers, depending on the quality of the index. If the index is of high quality, they offer a contract that calls for the borrower to repay 8/3 dollars if the realization of the index indicates the good state and nothing otherwise. If the index is of low quality, they offer a contract that calls for the borrower to repay one dollar regardless of the realization of the index. These offers constitute what we call a full-information optimal contracts equilibrium. To see why they can arise in equilibrium, note that all lenders earn weakly positive profits and could not possibly earn more by making different offers. Moreover, given that all lenders have common information, the borrower can perfectly infer the quality of the index by observing the contracts that the lenders offer. In other words, it is not possible for a single lender to convince the borrower that the index is of high quality if all of the other lenders offer a non-contingent contract. This same intuition carries over to the second type of equilibrium we describe, that is, the noncontingent contracts equilibrium.

Now suppose that all lenders offer a contract that calls for the borrower to repay one dollar regardless of whether the index is of high or low quality. These offers constitute what we call the noncontingent contracts equilibrium. Can a single lender gain by deviating and offering the best contingent contract? Again, the answer is no. If a single lender deviates by offering a contingent contract, then she will have to charge a premium for it to at least break even, where by premium, we mean that the contract calls for the borrower to repay an amount that in expectation exceeds the amount financed. In the case of the best contingent contract, the expected repayment of the borrower is 4/3, that is, 8/3 (the repayment if the index is in a good state) times 1/2 (the probability that the
index is in a good state), while the amount financed is one, so that the premium is 1/3. If the index is of low quality, lenders are risk-neutral with respect to the index and the premium is pure profit. If the index is of high quality, the premium is compensation for risk and leaves the lenders with zero net present value. Consequently, a lender would be at least as willing to make this offer given a low quality index as when given a high quality index. As such, standard belief refinements imply that the borrower can believe that the index is low quality after observing this deviation. Given these beliefs, the borrower is strictly better off when accepting one of the offers of a non-contingent contract. The failure of the agents to share risk in this case is closely related to the classic lemons market breakdown of Akerlof (1970).

Two features are essential to the existence of the noncontingent contracts equilibrium in this simple example. First, lenders are risk-averse with respect to expected payoffs across external states, and second, borrowers are even more risk-averse, meaning that it is efficient for the lender to insure the borrower. The first feature means that deviating from the noncontingent contracts equilibrium requires that a lender charge a premium for a contingent contract, which makes such a deviation more attractive when the index is of low quality. We discuss this example and the two key conditions in Section I.

Our general model encompasses settings in which there is an additional security design problem concerning payoffs given idiosyncratic states. These security design problems are important for our results in that they determine the borrower’s and lenders’ indirect utility over securities and external states, and thus the potential gains from indexation. In our mortgage example (Section II), the borrower needs incentives to repay the lender across idiosyncratic states. In this example, conditional on a particular external state, standard debt contracts are optimal. In principle, these debt contracts could allow for risk-sharing over the external states by having a higher face value in a good external state than a bad one. However, the face value of a debt contract is not equivalent to its expected payoff, that is, promises are not payoffs. A lender can prefer a higher debt level in a good external state simply because the debt is more likely to be repaid in good external states. At the same time, the lender has a lower marginal utility in the good external state. The key condition to generate a noncontingent contracts equilibrium becomes a tradeoff between the lender’s decreasing marginal utility and the increasing value of promises as the external state improves. If the latter force dominates, then the lender will not need to charge a premium to insure the borrower against bad external states and the noncontingent contracts equilibrium will not exist. A key condition for the existence of the noncontingent contracts equilibrium is that the lender be sufficiently “risk-averse over promises,” a notion that we formalize in our general model (Sections III, IV, and V).

There is an important distinction between the type of adverse selection problem we consider and one in which lenders have information about the external state itself. In our model, lenders do not have better information about the distribution of the external state; they only have information about the relationship between the index and the external state. In contrast, much of the literature on adverse selection (following Akerlof (1970)) assumes that there is asymmetric information about something that is directly relevant to payoffs. For example, in the context of mortgages, lenders may
know that local house prices are more likely to appreciate in the future than the borrower expects. In an extension of our model (Sectopm VI), we show that under our assumptions, the noncontingent contracts equilibrium does not exist if the index is known to be perfectly correlated with the external state, and the adverse selection is only about the distribution of the external state itself.

Our work is related to the literature on incomplete contracts, as surveyed by Tirole (1999). Papers focusing on incomplete contracts and asymmetric information include Spier (1992), Allen and Gale (1992), and Aghion and Hermalin (1990), among others.² Our model differs from most of this literature in several respects. First, our model emphasizes competitive markets, rather than bilateral negotiation. Second, our model focuses on asymmetric information about the quality of the index, rather than the “fundamentals.” This second difference allows us to generate noncontingent contracts in equilibrium without relying on transaction costs of using the index or arguing that the index is manipulable. Like some, but not all, of the incomplete contracts literature, we focus on equilibria with no indexation at all (as opposed to explaining why agents might use the index but not achieve perfect risk-sharing).

More significantly, our model differs from the incomplete contracts literature in its assumptions about what is contractible and what is observable. In the risk-sharing extension of Hart and Moore (1988), agents can renegotiate after observing a nonverifiable state. The subsequent literature (Green and Laffont (1992), Dewatripont and Maskin (1995), Segal and Whinston (2002)) shows that, by altering the outside options or other aspects of the renegotiation process, the agents can share risks and perhaps even achieve first-best risk sharing, despite their inability to contract on the state. In contrast, in our model the index is both observable and verifiable, whereas the true external state is not observed by the agents until the end of the game, when renegotiation is no longer possible.³

Formally, our model is similar in some respects to Allen and Gale (1992), although authors of that paper focus on the manipulability of the index. One can also view our model as related to models of insurance, in the vein of Rothschild and Stiglitz (1976) or more recently Hendren (2013). The key difference between our model and these models is that our model places the information advantage and the competition on the same side of the market (with lenders), rather than on opposite sides of the market. Loosely speaking, the key intuition in our model is that the insurance itself might be a “lemon,” in the sense of Akerlof (1970).

In a closely related paper, Spier (1992) shows that asymmetric information can amplify the effect of transaction costs on the ability of agents to write contracts that condition on relevant information. In Spier (1992), an informed and risk-averse principal contracts with an uninformed and risk-neutral agent. If the principal offers a contract that insulates herself from risk, then she must also signal her private information, which in turn reduces the benefits of risk-sharing. This effect lowers the level of transaction costs needed to destroy risk-sharing in equilibrium. However, in Spier (1992),

³Similarly, the Maskin and Tirole (1999) critique of the incomplete contracts literature applies when the agents are aware of the payoff-relevant states before actions are taken and, for this and other reasons, is not directly applicable to our model.
if transaction costs are close enough to zero, then asymmetric information alone does not eliminate risk sharing. In contrast, in our model, asymmetric information can lead to zero risk-sharing without transaction costs. In another related paper, Asriyan (2015) shows that concern for future liquidity and dispersed private information can lead market participants to write very simple contracts. The intuition here is that if the holder of a contract must liquidate at some future date, then she will want to hold a contract that is as informationally insensitive as possible. In contrast, we emphasize situations in which risk-sharing failures are associated with simple contracts. In other words, the value of simple contracts is informationally sensitive in our model, and only by using the index could the agents minimize information sensitivity.

We also employ a general space of states and contracts. As a result, there is a great deal of scope for signaling, in contrast to the previous literature (in Spier (1992) and Aghion and Hermalin (1990), the contract space has one or two dimensions). As a result of this ability to signal, to generate our results, borrowers must be somewhat “suspicious” in the sense that they place nonzero probability on the index being irrelevant. Belief in this possibility, however unlikely, creates at least some chance that the index is not useful (and in this sense is reminiscent of the conditions of the Myerson and Satterthwaite (1983) theorem).

The failure of risk-sharing in our model can be thought of as a coordination failure, in the sense that there are multiple, Pareto-ranked equilibria. In the context of mortgages, we view this multiplicity as a feature. Mortgage contracts differ substantially across countries in ways that are difficult to explain with “fundamentals.” Our model considers only a single index but could naturally be extended to consider multiple indices (interest rates and house prices, for example). In this case, we expect “partial indexing” equilibria exist (e.g., indexing to interest rates but not home prices, like an adjustable rate mortgage). As a result of this multiplicity, there is a potential for policy to improve welfare in our model by ruling out undesirable equilibria. However, our model does not feature any externalities as a result of this risk-sharing failure. The existence of such externalities (which are emphasized by Campbell, Giglio, and Pathak (2011), among others) would provide an additional motivation for policy interventions.

Our motivating example is the mortgage market, although our most general model is abstract and could easily apply to other settings. In the context of home ownership, as noted by Sinai and Souleles (2005), purchasing a house hedges a homeowner against changes in future rents. Nevertheless, homeowners are exposed to both price and rent risks, and these could be hedged through the mortgage contract. Of course, as discussed by Case, Shiller, and Weiss (1995) and Shiller (2008), homeowners could also hedge these risks through other financial markets, although this almost never occurs in practice. This failure to hedge might be explained by limited access to these markets or by the sophistication required to hedge in this manner. However, these arguments suggest that it would be profitable for a financial intermediary to provide hedging services. Mortgage lenders appear to be ideally situated to do this as part of their mortgage contracts.

Mortgages that have a more equity-like claim on house value have been proposed (see, for example, Caplin, Carr, Pollock, Yi Tong, Tan, and Thampy (2007)). Some of these early proposals made
mortgage payments contingent on the sale price of the house, which clearly induces moral hazard for the borrower. More recent studies point out that conditioning mortgage payments on an index of house prices avoids this problem. Relatedly, Campbell, Clara, and Cocco (2018) propose that mortgages that provide optional payment reductions during recessions increase financial stability. Piskorski and Tchistyi (2017) develop an equilibrium model of housing and mortgage markets. They show that under many circumstances, the optimal mortgage design hedges the borrower against house price risk. Greenwald, Landvoigt, and Van Nieuwerburgh (2018) provides a quantitative analysis of the general equilibrium effects of house-price-indexed mortgages and show that using a local house price index improves financial stability. Proposals for mortgage reform after the recent financial crisis (e.g., Mian and Sufi (2015)) have advocated this approach. Although rare, shared appreciation mortgages are legal in the U.S. and used, for example, by Stanford University faculty who borrow from Stanford to purchase a house.\footnote{Stanford mortgages are indexed to an appraisal rather than a local house price index, and involve renegotiation when the homeowner makes major investments.} We develop a stylized model of mortgage borrowing and show that the conditions of our general theorem apply in this model and thus can explain the lack of prevalence of shared appreciation mortgages by appealing to asymmetric information over the quality of house price indices.

The rest of this paper is structured as follows. We begin in Section I by using the numerical example outlined above to illustrate the two key assumptions required for noncontingent contracts to be an equilibrium. Next, in Section II, we discuss a more complicated example that focuses on mortgages, and we show how these key assumptions must be adapted in a setting in which default is possible. We then begin to describe our general model, discussing the market for loans, the asymmetric information problem, and the equilibrium concept in Section III. In Section IV, we discuss the zero-profit condition that arises from competition in our model and characterize the “best” equilibria, which features contingent contracts. In Section V, we discuss our most general results, which describe assumptions under which risk-sharing fails and noncontingent contracts arise in equilibrium. In Section VI, we describe a number of variations and extensions to our basic framework. We conclude the paper in Section VII.

I. Noncontingent Equilibrium: A Simple Example

We begin by elaborating on the example in our introduction. In this example, there are two possible “external” states: bad and good. We refer to these states as external to emphasize that they are outside of the control of the borrower and lenders. Let \( A = \{a_b, a_g\} \) be the set of possible external states. There are also two possible index realizations, high and low. Let \( Z = \{z_l, z_h\} \) be the set of possible index realizations. The two external states are equally likely, \( P(a = a_b) = P(a = a_g) = 1/2 \), as are the two index values, \( P(z = z_l) = P(z = z_h) = 1/2 \). The index might be “perfect,” in which case \( z = z_h \) if and only if \( a = a_g \), and \( z = z_l \) if and only if \( a = a_b \). The index might also be “uninformative,” in which case the realizations of \( z \in Z \) and \( a \in A \) are independent.
Recall from our example that the borrower’s marginal value of a dollar is 1/2 if \( a = a_g \) and 3/2 if \( a = a_b \), whereas the lenders’ marginal value of a dollar is 3/4 if \( a = a_g \) and 5/4 if \( a = a_b \). We considered two contracts, a contingent (on \( z \in Z \)) contract and a noncontingent contract. The contingent contract requires the borrower to pay \( d = 8/3 \) if \( z = z_h \) and \( d = 0 \) if \( z = z_l \), whereas the noncontingent contract requires the borrower to pay \( d = 1 \) regardless of the value of \( z \).

The initial investment required is \( K = 1 \). As a result, the noncontingent contract is break-even for the lenders, regardless of whether the index is perfect or uninformative. The contingent contract is break-even for the lenders if the index is perfect. However, the contingent contract is positive net present value for the lenders if the index is uninformative. As a result, a lender has the greatest incentive to deviate from a noncontingent equilibrium when the index is uninformative.

We conclude that using the noncontingent contract is an equilibrium. If the borrower expected the noncontingent contract but was offered the contingent contract instead, then she could (and perhaps should) assume that the index is uninformative because a lender with an uninformative index has the most to gain by deviating to the contingent contract. In this case, the lender’s gain is the borrower’s loss because the contingent contract does not provide insurance that is useful to the borrower and thus the borrower should reject this deviation.

This conclusion depends on two key assumptions. First, the borrower and the lenders agree on which states have high/low marginal values of a dollar. If instead the lenders’ marginal value of a dollar were 5/4 if \( a = a_g \) and 3/4 if \( a = a_b \) (reversing the order for the lenders), then the contingent contract that breaks even for the lenders given a perfect index is \( d = 8/5 \) if \( z = z_h \) and \( d = 0 \) if \( z = z_l \). However, a lender with an uninformative index would not offer this contract because the amount financed exceeds the amount that the borrower repays in expectation. As a result, the non-contingent equilibrium could not exist. We conclude that agreement about which states have high/low marginal values is critical for the existence of a noncontingent equilibrium.

The second key assumption is that the borrower should buy insurance from the lenders, and not vice versa. Suppose that we switch the marginal values between the borrower and the lenders, so that the borrower’s marginal value of a dollar is 3/4 if \( a = a_g \) and 5/4 if \( a = a_b \), and the lenders’ marginal value of a dollar is 1/2 if \( a = a_g \) and 3/2 if \( a = a_b \). In this case, the best contingent contract that breaks even for the lenders given a perfect index is \( d = 0 \) if \( z = z_h \) and \( d = 4/3 \) if \( z = z_l \). In other words, the lenders buy insurance from the borrower using this contingent contract. We again find that a lender with an uninformative index would not offer this contract because the amount financed exceeds the amount that the borrower repays in expectation. As a result, the noncontingent equilibrium could not exist. We conclude that the borrower must be “more sensitive” to the external state \( a \) than the lenders, in the sense that the lenders should be insuring the borrower and not vice versa, for a noncontingent equilibrium to exist.

Looking ahead, these two assumptions correspond exactly to the key assumptions in our general model. Intuitively, if the lenders can insure the borrower at a negative insurance premium (i.e., with a contract that calls for the borrower to repay an amount that in expectation is less than the amount financed), or if the lenders should be buying insurance from the borrower and can offer a high price
for that insurance, a lender can prove that the index is perfect. But if the lenders should be insuring the buyer and requires a (weakly) positive insurance premium to do so, then the noncontingent equilibrium can exist. Moreover, the noncontingent equilibrium exists even though, if the index is in fact perfect, both the lenders and the borrower can be made better off by using the index.

This example is constructed to clearly illustrate the key requirements for the existence of a non-contingent equilibrium. The example is simple because both the borrower and the lenders have constant marginal utility within each external state $a$. While this might make sense for a lender (e.g., if the stochastic discount factor (SDF) of the lenders is a function of $a$, and this transaction is small relative to the size of a lender), it often does not make sense as a model of borrowers. In the context of mortgages, we would prefer to assume that borrowers have concave utility. We would also like to incorporate the possibility that the borrower defaults instead of repaying the promised value $d$. For this reason, before describing our general model, we introduce a simple model of the mortgage market. We also show how the two key assumptions of “agreement on marginal values” and “lenders should insure the borrower” can be understood in this context.

II. Noncontingent Equilibrium: A Mortgage Example

In this section, we discuss a simple model of mortgage lending that illustrates the key conditions necessary for the existence of a non-contingent contracts equilibrium.

A. The Setup

There are two dates. At date $t = 0$, a borrower receives take-it-or-leave-it offers from $|L| \geq 3$ mortgage lenders to finance the purchase of a house for $K$ dollars. The borrower promises repayment at date $t = 1$, collateralized by the house. The borrower can accept one offer and occupies the house until the start of date $t = 1$, at which point she liquidates it and consumes her final wealth. The value of the house at date $t = 1$ is $x \in \{0, x_h\}$ with $K < x_h$. As in the previous example, there are two external states, $A = \{a_b, a_g\}$, and two possible index values, $Z = \{z_l, z_h\}$. The external state $a \in A$ affects the likelihood of a high house price and both agents’ other sources of income. The external state could represent the aggregate component of house prices in a local area containing the borrower’s house or a broader economic variable that affects house prices and the agent’s other sources of income. The external state could therefore be thought of as either a house price index or a broader economic index.

The realization of the index $z$ is publicly observable, while the realizations of the house price $x$ and external state $a$ are not. The marginal distributions of $a$ and $z$ are common knowledge, and for simplicity we assume as in the previous example that $P(a = a_g) = P(z = z_h) = 1/2$. The joint distribution of $x$ and $a$, $P(x = x_h | a) = \pi(a)$, is also common knowledge, with high house prices being more likely in good states, $\pi(a_b) < \pi(a_g)$.

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5The assumption that the liquidation value is either high or zero simplifies the exposition considerably.

6This example is too simple to draw a distinction between these two types of indices.
The lenders all privately observe the joint distribution of \(a\) and \(z\), given by \(\theta(a, z)\), where

\[
\begin{align*}
\theta(a_g, z_h) &= \theta(a_b, z_i) = \frac{1}{4} + \rho, \\
\theta(a_g, z_i) &= \theta(a_b, z_h) = \frac{1}{4} - \rho,
\end{align*}
\]

for some \(\rho \in [0, \bar{\rho}]\).\(^7\) We refer to \(\rho\) as the quality of the index. The borrower does not know \(\rho\) and has a prior with full support on \(\rho \in [0, \bar{\rho}]\). After the borrower observes the contracts offered by the various lenders, she may update her beliefs about \(\rho\).

To motivate the use of debt (which can be noncontingent or contingent on the index \(z\)), we assume that the borrower privately observes the value of her house at date \(t = 1\) and can make a report \(\bar{x}\) to her lender. While the lenders cannot observe the value of the house, they can implement a foreclosure rule conditional on the borrower’s report. For simplicity, we assume that foreclosure results in liquidation value of \(x = 0\) and that the lenders cannot randomize foreclosure.\(^8\) As a result, any feasible outcome can be implemented by debt with face value \(d(z)\), which depends on the index realization \(z\), such that if the borrower fails to repay the face value, then the lender forecloses on the house. We assume in what follows that \(d(z) \leq x_h\), and therefore the borrower will default if \(x = 0\) and repay if \(x = x_h\).

In addition to her house, the borrower has a nonpledgeable endowment \(y_B(a)\), with \(y_B(a_b) < y_B(a_g)\), available at date \(t = 1\). Consequently, the borrower’s final wealth is \(W = y_B(a) + x_h - d\) if \(x = x_h\) and the borrower owes \(d\) to the lender that makes the loan, and \(W = y_B(a)\) otherwise. The borrower has constant relative risk aversion (CRRA) expected utility \(E[u_B(W)]\) over final wealth, where

\[
u_B(W) = \frac{W^{1-\gamma_B} - 1}{1-\gamma_B}.
\]

We summarize the payoffs for a borrower that owes \(d\) in external state \(a\) using the indirect utility function

\[
\phi_B(d, a) = \pi(a)u_B(y_B(a) + x_h - d) + (1 - \pi(a))u_B(y_B(a)).
\]

If the borrower does not purchase the house, then she receives \(u_B(y_B(a))\) in external state \(a\). For simplicity, we assume that defaulting and never purchasing the house are identical from the borrower’s perspective, so the borrower’s participation constraint will never bind.

All lenders have an existing portfolio of assets (e.g., other mortgages) with value \(y_L(a)\), again with \(y_L(a_b) < y_L(a_g)\). Conditional on making the loan, a lender’s total asset value is \(R = y_L(a) + d - K\) if the borrower repays \(d\), and \(R = y_L(a) - K\) otherwise. Lenders derive CRRA expected utility \(E[u_L(R)]\) over their total payoff \(R\), where

\[
u_L(R) = \frac{R^{1-\gamma_L} - 1}{1-\gamma_L}.
\]

We assume that this particular borrower is small relative to the lenders, which means that \(d\) and

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\(^7\)If \(\bar{\rho} = \frac{1}{4}\), a perfect index is possible. However, our assumptions require only that \(\rho > 0\).

\(^8\)This is a simplified costly state verification model (Townsend (1979), Gale and Hellwig (1985)).
$K$ are small relative to $y_L(a)$. We also normalize the lenders’ indirect utility function, $\phi_L(d, a)$, by subtracting the expected utility if the lender does not make a loan to this borrower, so that lenders’ indirect utility is zero if they do not make the loan. As a result, using a first-order Taylor expansion, if a lender is owed $d$ in external state $a$, then the lender has indirect utility

$$\phi_L(d, a) = \pi(a)u'_L(y_L(a))(d - K) - (1 - \pi(a))u'_L(y_L(a))K.$$  

Note that $u'_L(y_L(a))$ can be interpreted as an SDF.

We now impose assumptions on the endowments and preferences of the lenders and the borrower that ensure agreement on marginal values and that imply the lenders should insure the borrower. First, we assume that lenders’ endowment and risk aversion satisfy

$$\gamma_L \log \left( \frac{y_L(a_g)}{y_L(a_b)} \right) \geq \log \left( \frac{\pi(a_g)}{\pi(a_b)} \right),$$ \hspace{0.5cm} (1)$$

and the borrower’s endowment and risk aversion satisfy

$$\gamma_B \log \left( \frac{y_B(a_g) + x_h}{y_B(a_b) + x_h} \right) \geq \log \left( \frac{\pi(a_g)}{\pi(a_b)} \right).$$ \hspace{0.5cm} (2)$$

The conditions in equations (1) and (2) state that the percentage change in marginal utility between $a_b$ and $a_g$ is greater than the percentage change in the default probability. As a result, both the borrower and the lenders have decreasing (between $a_b$ and $a_g$) marginal utilities with respect to payments, which occur only if the borrower does not default. In other words, they agree that such payments have a high marginal value in external state $a_b$ and a low marginal value in $a_g$.

Next, we assume that the borrower faces a greater cost of bearing the risk of her endowment than the lenders, conditional on no default, that is,

$$\gamma_B \log \left( \frac{y_B(a_g) + x_h}{y_B(a_b) + x_h} \right) > \gamma_L \log \left( \frac{y_L(a_g)}{y_L(a_b)} \right).$$ \hspace{0.5cm} (3)$$

Equation (3) implies that under full information, it is efficient for the lenders to insure the borrower.\(^9\)

An equilibrium of this market is given by 1) a set of offers $d_l(z)$, for each lender $l$, such that lenders maximize their expected utility, 2) a borrower belief function that maps the set of possible offers to a posterior belief about the quality of the index and is consistent with Bayes’ rule where possible, and 3) an acceptance rule that maximizes the borrower’s utility conditional on her beliefs about the quality of the index. We describe the market structure and equilibrium definition in more detail in our general model.

### B. Analysis

Equations (1) and (2) ensure that both the borrower and the lenders agree that the marginal value of a promise (a payment conditional on no default) is higher in the bad external state. In other

\(^9\)Note that equations (3) and (1) imply equation (2). An analogous result appears in our general model.
words, they are both risk-averse with respect to promises. This kind of risk aversion is related but not identical to having a risk-averse utility function. For example, because the loan under consideration is small relative to a lender’s other income, the lenders are effectively risk-neutral with respect to the borrower’s idiosyncratic outcome. However, the lenders can still be risk-averse with respect to promises because the lenders’ SDF is a function of the external state \( a \in A \).

This type of risk aversion for the lenders can arise for at least three reasons. First, a lender can be thought of as another agent in the economy, with her own CRRA preferences and other sources of income. Second, suppose that the lenders are intermediaries subject to regulatory or financial constraints and that \( y_L \) is the cash flow on a lender’s portfolio of other mortgages.\(^{10}\) When the external state is good (bad) because the aggregate component of house prices is high (low), the cash flow from a lender’s mortgage portfolio is higher (lower) and repayment of one individual loan has a smaller (larger) effect on the health of a lender’s balance sheet. Third, suppose that the lenders are integrated with financial markets, in which case \( y_L \) is the representative agent’s consumption and \( \gamma_L \) is the representative agent’s relative risk aversion. In this case, the external state affects broader economic conditions. When the economy is in a good (bad) state, the lenders’ SDF is lower (higher).

Two competing forces determine whether the lenders are risk-averse with respect to promises. First, the lenders are risk-averse with respect to the external state, which means that the lenders have higher marginal utility in the bad external state. Second, promises are more likely to be paid in the good external state \((\pi(a_h) > \pi(a_l))\). Equation (1) implies that the first of these forces weakly dominates the second, so that the lenders have a higher marginal value of promises in bad states, which means that

\[
\frac{\partial}{\partial d} \phi_L(d, a) = \pi(a)u_L'(y_L(a))
\]  

(4)

is decreasing in \( a \). Note that equation (1) is equivalent to assuming that the lenders’ SDF is more volatile than mortgage default probabilities.

As with the lenders, two competing forces determine whether the borrower is risk-averse with respect to promises. The borrower is risk-averse, and hence will have higher marginal utility in the bad external state. At the same time, in the bad state, promises are less likely to be repaid and hence are less costly to make. Equation (2) implies that the first of these forces dominates the second, so that

\[
\frac{\partial}{\partial d} \phi_B(d, a) = -\pi(a)u_B'(y_B(a) + x_h - d)
\]  

(5)

is increasing in \( a \). Note that, unlike the lenders, the marginal cost of a promise for the borrower depends on the size of the promise. If the borrower is risk-averse with respect to promises \((\frac{\partial}{\partial d} \phi_B(d, a))\), which is negative, is increasing in \( a \) at \( d = 0 \), then she will be risk averse with respect to promises at all higher debt levels.\(^{11}\)

As discussed in Section I, although the borrower and the lenders agree on which external state

\(^{10}\)In this case, the CRRA preferences should be understood as a proxy for the curvature induced by the financial or regulatory constraints.

\(^{11}\)This follows from \( u_B''(\cdot) > 0 \), which we prove the proof of Proposition 1.
has a higher marginal benefit/cost of promises, this is not sufficient to guarantee the existence of a non-contingent equilibrium. We also need to ensure that the lenders should be insuring the borrower, and not vice versa. Intuitively, which agents should be providing insurance and which agents should be receiving insurance depends on the ratio of the marginal values of promises. Equation (3) implies that the borrower's marginal value of a promise is more sensitive to the external state than the lenders'. In particular, this property holds at 

\[
\frac{\partial}{\partial d} \phi_B(d, a_g)_{|d=0} < \frac{\partial}{\partial d} \phi_L(d, a_b)_{|d=0} < \frac{\partial}{\partial d} \phi_L(d, a_b)_{|d=0}.
\]  

Equation (6) also implies that if the index is related to the external state (\(\rho > 0\)), then the first-best contract features promised face value payments that increase with the index, \(d(z_l) < d(z_h)\).

We now give a heuristic argument that the asymmetric information between the lenders and the borrower over the quality of the index can lead to the use of noncontingent contracts. Suppose that \(L - 1\) lenders offer the same noncontingent contract \(d^*\). Can the \(L\)-th lender offer a contingent contract \(d'\), with \(d'(z_l) < d'(z_h)\), to exploit the risk-sharing benefits that such contracts offer? If the borrower accepts the offer of \(d'\), the lower the index quality (lower \(\rho\)), the higher the profit for the lender. That is,

\[
\frac{\partial}{\partial \rho} \mathbb{E}[\phi_L(d'(z), a)] = (\phi_L(d'(z_h), a_g) - (\phi_L(d'(z_l), a_g)) - ((\phi_L(d'(z_h), a_b) - (\phi_L(d'(z_l), a_b))) < 0,
\]

by the definition of \(\rho\) and the lender’s risk aversion with respect to promises. Intuitively, the higher the index quality, the more variation in \(d'(z)\) is correlated with lenders’ endowment, and consequently the more costly it is for a lender to offer this insurance. Therefore, if the borrower is offered \(d'\), then it is reasonable for the borrower to believe that the index has the lowest quality. Given this belief, the borrower will reject any \(d'\) that the \(L\)-th lender would be willing to offer. What if the \(L\)-th lender offered a contingent contract that was decreasing in \(z\), \(d'(z_l) > d'(z_h)\)? In this case, the \(L\)-th lender is offering to purchase insurance from the borrower, which is inefficient. As a result, if a lender is willing to offer a decreasing contract, the borrower is not willing to accept it. This argument leads to the following proposition.

**PROPOSITION 1:** There exists a \(\bar{K} > 0\) such that, for all \(K < \bar{K}\), there exists an equilibrium in which non-contingent contracts are used regardless of the type \(\theta\).

**Proof:** See Appendix B.

The requirement that \(K < \bar{K}\) is needed to guarantee, among other things, that is possible to finance the house with a noncontingent contract.\(^{12}\)

In this mortgage example, we have illustrated the distinction between promises and payments,

\(^{12}\)We prove Proposition 1 by first demonstrating that the mortgage model of this section satisfies the assumptions of our general model (described in the next sections), provided that \(K < \bar{K}\) for some \(\bar{K} > 0\), and then invoking our most general result, Proposition 3 below. The assumptions of our general model are sufficient but not necessary. We speculate that a proof tailored to this mortgage example could prove Proposition 1 under weaker conditions (a higher value of \(\bar{K}\)).
and what it means to be risk-averse with respect to promises. We now turn to the general version of
the model, which treats the indirect utility functions $\phi_B$ and $\phi_L$ as primitives. We provide a set of
assumptions (Assumptions 2 and 3 below) that generalize equations (1) and (3). These assumptions
ensure that both the lenders and the borrower are risk-averse with respect to promises, and that
it is efficient for a lender to insure the borrower. We then prove a general result, Proposition 3,
which shows that as long as these assumptions are satisfied, there exists an equilibrium with non-
contingent contracts. The model of mortgage lending that we describe in this section satisfies the
assumptions of our general model, meaning that Proposition 1 is a consequence of our general result
in Proposition 3.

III. The General Model

In this section, we describe our general model. At date $t = 0$, a borrower wishes to raise $K > 0$
dollars to pursue a project (e.g., purchase a home). If the borrower accepts a contract offer by a
lender, then the borrower will initiate the project. Payoffs are determined at date $t = 1$.

We now describe each component of the model in more detail. First, we introduce the external
states $a \in A$ and index $z \in Z$, which are the key exogenous random variables in the model. Second, we
describe the contracting environment and the indirect utility functions that summarize the payoffs
of the borrower and lender from using a particular contract. Third, we introduce the “types” in our
model, which describe index quality, that is, the relationship between the external state and the
index. Fourth, we discuss the market structure. Finally, we define the equilibrium in the context of
our model.

A. The States and the Index

After the borrower and a lender agree to a contract and initiate the project, an index $z \in Z$ is
determined at date $t = 1$. This index is observable and verifiable, and related to the true external
state $a \in A$. The true external state $a$ is what enters the agents’ indirect utility functions—they have
no particular concern over the value of the index.

The index $z \in Z$ should be thought of as an index based on the external state $a \in A$. For simplicity,
we assume that both $A$ and $Z$ are totally ordered sets. We write $a > a'$ to capture the idea that
the external state $a \in A$ is “better than” the external state $a' \in A$, and we use the same notation
for the index values. In the context of mortgages, the external state $a \in A$ might influence house
prices, borrower income, and/or the lenders’ cost of capital. The index $z \in Z$ is an index that perhaps
imperfectly measures these things, such as a local area house price index, a wage index, or an interest
rate. We assume that $A$ and $Z$ are finite sets.

The external state $a \in A$ influences the distribution of the borrower’s idiosyncratic outcomes, $i \in I$.
For a mortgage borrower, idiosyncratic outcomes could include the borrower’s particular house price
or income. The idiosyncratic outcomes may or may not be observable or contractible, and might be
influenced by the borrower’s behavior.
B. The Indirect Utility Functions and Contract Space

A contract is a function \( s : I \times Z \rightarrow \mathbb{R}^+ \) maps the idiosyncratic outcome \( i \) and index \( z \) to a payment from the borrower to the lender. We use the notation \( s_z : I \rightarrow \mathbb{R}^+ \) to refer to the “conditional contract,” which is the contract for a particular value of the index.

We require that conditional contracts be “ex-post efficient,” appealing to notions of renegotiation-proofness after the index \( z \in Z \) has been revealed. We assume that the set of ex-post efficient contracts, which we denote by \( S_D \), can be indexed by a number, \( d \in D \subset \mathbb{R} \). We further assume that \( D \) is a convex subset of the real line whose minimum is \( d = 0 \).\(^{13}\) For example, if the ex-post efficient contract is a debt contract, as in many optimal contracting models (e.g., Hart and Moore (1998), Innes (1990), Townsend (1979), Hébert (2018)), then \( d \) is the face value of the debt claim. For this reason, we refer to the parameter \( d \) as a “promise.” We use the notation \( s_d \in S_D \) to indicate an ex-post efficient contract with a promise of \( d \). The set of feasible contracts \( S \) is the set of contracts such that for each \( z \in Z, s_z \in S_D \).

The primitives of our model are the indirect utility functions of the borrower and lender. Given a particular state \( a \in A \) and ex-post efficient contract \( s_d \),\(^{14}\) the borrower’s indirect utility function is \( \phi_B(s_d, a) \). We refer to this as an indirect utility function because it summarizes the borrower’s payoff, given some underlying relationship between the external state \( a \), the contract \( s_d \), and the distribution of idiosyncratic outcomes. Similarly, we denote the lender’s payoff (including the cost of the initial investment \( K \)) by \( \phi_L(s_d, a) \). These functions should be understood as expected utilities conditional on \( a \), and do not necessarily imply that the borrower or the lender perfectly knows \( a \).

We treat the indirect utility functions as primitives that satisfy several properties. First, we assume that \( \phi_L(s_d, a) \) is lower semi-continuous on \( d \in D \) for all \( a \in A \).\(^{15}\) Second, we assume that \( \phi_L(s_d, a) \) is negative for \( d = 0 \). We set the lender’s indirect utility to zero if she does not make the loan, so this property implies that the lender is not willing to make the loan in exchange for a promise of zero in all states. Third, the borrower’s utility function satisfies a monotonicity property: if \( d' > d \) for some \( d, d' \in D \), then \( \phi_B(s_d', a) \leq \phi_B(s_d, a) \) for all \( a \in A \). Intuitively, if the borrower makes a larger promise to the lender, then she is worse off. For the lender, this property does not necessarily hold; promises will not necessarily be paid, and demanding excessive repayment can result in lower expected utility for the lender.

Although we use debt as our leading example, these conditions can also describe other families of securities. Examples include the set of fixed payments of varying size, the set of 100% equity claims less a fixed payment of varying size, and the set of equity shares of varying percentages. The first two of these examples could be motivated by risk-sharing type problems, and the third by security design problems resulting in equity as the optimal security design.

\(^{13}\) In our mortgage example (Section II), \( d \) is the face value of the debt claim and it is natural to restrict attention to \( D \in [0, x_h] \).

\(^{14}\) These indirect utility functions are naturally defined over all conditional contracts, not just ex-post efficient ones, but ex-post inefficient contracts play no role in our analysis.

\(^{15}\) Inefficient foreclosure/liquidation can generate downward jumps in the lender’s payoff in certain cases, which is why we assume lower semi-continuity instead of continuity.
Our simple example (Section I) and mortgage example (Section II) provide examples of indirect utility functions $\phi_L$ and $\phi_B$ that satisfy our assumptions. In Appendix A, we provide an example that is based on costly state verification (CSV) models.

C. Types

We define $\theta(a,z)$ as the joint distribution of the external state and the index and refer to it as the lenders’ (common) “type.” This joint distribution is common knowledge among the lenders but is not known to the borrower. The lenders’ type $\theta$ is drawn from a set $\Theta$, the set of all joint distributions that have the same marginal distributions for $a \in A$ and $z \in Z$, which we denote by $p(a)$ and $q(z)$, respectively. Without loss of generality, we assume that these marginal distributions have full support over $A$ and $Z$, respectively. Let $\theta_0(a,z) = p(a)q(z)$ denote an “uninformative type,” that is, the type with an index that is independent of the external state.

The borrower’s prior belief over these types is $\mu_0$. In effect, the borrower is uncertain about the relationship between the index and the external state. A homeowner, for example, might not be sure how the S&P Case-Shiller index for his metro area relates to the price of his particular house. We assume that the borrower is aware of the marginal distributions, to abstract from the problems generated by that type of asymmetric information, and focus instead on the borrower’s uncertainty about the relevance of the index (we revisit this in our extensions in Section VI). We do not require that the beliefs $\mu_0$ have full support on $\Theta$, but rather impose assumptions on the support, which we describe below. In particular, we do not require that the type space contains a perfectly informative index; our results continue to hold, however, if such a type exists.

Having defined the type space, we next describe the market for loans.

D. The Market for Loans

Let $L$ denote the set of lenders, where $|L| \geq 3$, each of whom can post a contract. After these lenders post contracts, the borrower can pick whichever one she prefers or choose to forgo the investment opportunity. The outside options for the lenders are normalized to zero. Note that from the borrower’s perspective, lenders are perfect substitutes.

Let $S_L = (s^1, s^2, \ldots, s^{|L|})$ be the menu of contracts offered by the lenders at date $t = 0$. From lender $l$’s perspective, the expected utility of offering a contract $s^l \in S$ — when the other lenders offer contracts $S^{-l}$, the resulting menu is $S_L = (s^l, S^{-l})$, and the common type is $\theta$ — is

$$\sigma(s^l, S_L) \sum_{a \in A, z \in Z} \theta(a,z) \phi_L(s^l_z, a),$$

(7)

where $\sigma(s^l, S_L)$ is the probability that the buyer accepts the contract $s^l$ given the menu of contracts posted. This notation implicitly assumes that the buyer’s decision does not depend on the identity of the lender but rather on the contract that the lender offers. We maintain this assumption in the equilibria we study, and note that it is consistent with the assumption that the borrower’s utility does not depend on the lender she chooses, but rather on the design of the contract.
Assuming that the borrower chooses to borrow, her expected payoff for contract \( s \) is (abusing summation notation)
\[
\sum_{\theta' \in \Theta, a \in A, z \in Z} \mu(\theta'; S^L) \theta'(a, z) \phi_B(s_z, a),
\]
where \( \mu(\theta'; S^L) \) denotes the borrower’s beliefs about the distribution of the lender’s common type \( \theta' \) after observing the menu \( S^L \). The beliefs \( \mu(\theta'; S^L) \) are central to our theory. The borrower does not observe the lender’s common type \( \theta \). Initially, she has prior \( \mu_0 \) over the set of types \( \Theta \), but might update these beliefs based on the menu of securities offered. It is important to note that, because the type \( \theta \) is common across lenders, an optimal mechanism could allow the borrower to solicit this information and then negotiate a contract (Cremer and McLean (1988)). The market structure that we impose, which we believe is realistic in many contexts, prevents the buyer from conducting this sort of auction.\(^{16}\)

Having discussed the basic structure of the model, we next describe the equilibrium concept and the refinements for off-equilibrium beliefs that we employ.

**E. Equilibrium Definition**

The basic equilibrium concept that we use is perfect Bayesian, and we focus on pure strategies for the lenders. Given the strategies of the other lenders \( (S^{-l}_* \) and the borrower \( (\sigma^* \), and the common type \( \theta \), we require that lender \( l \) post

\[
s^l \in \text{argmax}_{s \in S} \sigma^*(s, S^{-l} \) \sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s_z^l, a),
\]

if the strategy \( s^l \) yields weakly positive expected utility, and otherwise does not participate. That is, each lender’s choice of contract maximizes his utility, given the strategies of the other lenders and borrower.

If the borrower is offered one or more contracts, she must choose a strategy \( \sigma(s^l, S^L) \) such that, given posterior beliefs \( \mu(\cdot; S^L) \), if \( \sigma(s^l, S^L) > 0 \), then

\[
s^l \in \text{argmax}_{s \in S_L} \sum_{\theta' \in \Theta, a \in A, z \in Z} \mu(\theta'; S^L) \theta'(a, z) \phi_B(s_z^l, a)
\]

\[
\text{and}
\sum_{\theta' \in \Theta, a \in A, z \in Z} \mu(\theta'; S^L) \theta'(a, z) \phi_B(s_z^l, a) \geq \bar{\phi}_B,
\]

where \( \bar{\phi}_B \) denotes the borrower’s expected utility if she does not accept any contract. In short, the borrower must maximize her utility given the menu of contracts being offered.

The equilibrium strategies of the lenders lead to a function \( S^*(\theta) \) that describes the menu of securities that might be offered, given the common type. If the borrower observes a menu \( S^L \) for

\(^{16}\)The mechanism of Cremer and McLean (1988) also requires commitment, and hence is inconsistent with our ex-post efficiency assumption.
which there exists a type $\theta'$ such that $S^L = S^*(\theta')$, then she must update her beliefs according to Bayes’ rule:

$$
\mu(\theta; S^L) = \frac{\mu_0(\theta)1(S^L = S^*(\theta))}{\sum_{\theta' \in \Theta} \mu_0(\theta')1(S^L = S^*(\theta'))}.
$$

(12)

This does not, of course, pin down what the borrower believes when she observes a menu $S^L$ that could not have been generated from the equilibrium strategies $S^*(\theta)$, for any $\theta \in \Theta$ with $\mu_0(\theta) > 0$. To determine whether a conjectured set of strategies is an equilibrium, we need only consider menus $S^L$ that differ from a menu $S^*(\theta')$ for a single lender. The result we are building towards is that there are many equilibria. This would be expected in the absence of refinements for off-equilibrium beliefs. Without refinements, however, the borrower can in effect dictate the contract by forming pessimistic beliefs when offered any other contract, justifying rejection. This kind of multiplicity is common in multi-sender signaling games (in our context, each lender is sending a signal about the common type). We therefore employ two refinements — unprejudiced beliefs and $D1$ — in line with common practice in the literature on multi-sender signaling games (Vida and Honryo (2019)).

The first refinement, unprejudiced beliefs (Bagwell and Ramey (1991)), requires that the borrower believe that the minimal number of lenders have deviated from equilibrium play. For concreteness, suppose that the true common type is $\theta$, all but one of the lenders offer an equilibrium contract for that type, and the remaining lender deviates by offering a security that is not offered by type $\theta$ in equilibrium. Moreover, suppose that the resulting menu could not have arisen from the equilibrium strategies of any type. Absent the refinement of unprejudiced beliefs the borrower could believe that multiple lenders have deviated. However, by imposing this refinement and using the fact that there are at least three lenders, the borrower must instead correctly identify the deviating lender.

The second refinement we that employ is the $D1$ equilibrium refinement (Banks and Sobel (1987)), which is applied to the correctly identified deviating lender. This refinement captures the idea that, if confronted with a “deviating” contract, the borrower should believe that the lender is of a type that would benefit from this deviation. Under our first refinement, the borrower is able to identify the deviating lender (when there is only a single deviating lender). We then apply the $D1$ refinement to the security offered by this lender. We believe that our results are robust to using refinements other than $D1$ that provide a similar intuition.

Following the standard definition of $D1$, we think of the borrower’s strategy as consisting of an acceptance probability $\xi$. A lender of type $\theta$ offering contract $s'$, instead of the equilibrium contract

\[17\] As discussed by Vida and Honryo (2019), it is common in multi-sender signaling games to require that equilibria satisfy both unprejudiced beliefs and the intuitive criterion of Cho and Kreps (1987). Unprejudiced beliefs allows the receiver to identify the deviating sender, and the intuitive criterion, usually applied in single-sender games, is applied to this deviating sender. For an example, see Schultz (1996). In our setting, the intuitive criterion has no bite, but $D1$, which is a similar but stronger refinement, does. As noted by Vida and Honryo (2019), unprejudiced beliefs, the intuitive criterion, and $D1$ are all related to strategic stability (Kohlberg and Mertens (1986)). However, we are not aware of other papers that have applied unprejudiced beliefs and $D1$ simultaneously.
s, benefits from this deviation if

\begin{equation}
\xi \sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s_z', a) \geq \sigma^*(s, S^*(\theta)) \sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s_z, a).
\end{equation}

The types for whom the set of $\xi \in [0,1]$ satisfying this condition is maximal are those on which the buyer's beliefs can place positive support following this deviation.

Looking ahead, we show below that in equilibrium, a lender's expected utility is equal to her outside option of zero due to the effects of competition. As a result, the $D1$ refinement will simply imply that the buyer must place the support of her beliefs on types that would weakly profit from offering the deviating contract, if that contract were accepted and such a type exists. The buyer cannot believe the deviating lender is of a type such that the lender would lose money if the buyer accepted the deviating contract, unless every lender type would lose money if the contract were accepted (but in this case, the deviating contract would never be offered). The requirement that the borrower should believe that the deviating lender would weakly profit if the borrower accepted the deviating contract is intuitive in our setting. We demonstrate that this requirement does not eliminate the noncontingent contract's equilibrium.

Our analysis below focuses on a particular set of equilibria, namely, symmetric pure-strategy equilibria. These equilibria are pure-strategy equilibria and symmetric in the sense that, for all types $\theta \in \Theta$, either all of the lenders offer the same security with certainty, $s(\theta)$, or none of the lenders offer a security. These equilibria are also symmetric in the sense that the borrower, when facing a menu of identical securities, chooses each lender with probability $|L|^{-1}$.

IV. Preliminary Analysis

We begin our analysis by focusing on the effects of competition. Consider a symmetric pure-strategy equilibrium and suppose that lenders' profits from offering the contract $s(\theta)$ are strictly positive. Intuitively, this could not be an equilibrium. Suppose that a lender offered a deviating contract $s' \in S$ such that for each index value $z \in Z$, the associated promise $d'_z$ was less than the promise associated with the original contract, $d_z$. The buyer would then be better off regardless of her beliefs, and therefore accept the contract with probability one. By sacrificing some profit, the lender would capture the entire market and be better off. Because of the monotonicity property of the buyer's indirect utility function and the lower semi-continuity property of the lender's indirect utility function, standard Bertrand competition effects apply and profits must be zero in equilibrium.

LEMMA 1: In any symmetric pure-strategy equilibrium, expected utility is zero for all lenders.

Proof: See Appendix B.

We next introduce an assumption to ensure that there exist contracts that can satisfy both the lenders' and the borrower's participation constraints.

ASSUMPTION 1: There exists a contract $s \in S_D$ that offers sufficient utility to the borrower, while
also satisfying the lender’s participation constraint, that is, the problem
\[
\max_{s \in S} \sum_{a \in A, z \in Z} \theta_0(a, z) \phi_B(s, a)
\]
subject to the constraint \( \sum_{a \in A, z \in Z} \theta_0(a, z) \phi_L(s, a) = 0 \) is feasible and has a solution that is weakly greater than the borrower’s outside option \( \bar{\phi}_B \).

Because we have assumed that the marginal distributions are the same for all types \( \theta \in \Theta \), this assumption is sufficient to ensure that for any type, there exists a contract that both the borrower and the lender would be willing to accept under full information.

Next, we turn to the existence of a “best” equilibrium. Consider a symmetric pure-strategy equilibrium, which is described by an offer of the contract \( s(\theta) \). Suppose that the mapping between types \( \theta \) and securities \( s(\theta) \) is one-to-one. In this case, in equilibrium the borrower knows the lenders’ common type. We then define a full-information optimal contract as
\[
\bar{s}(\theta) \in \arg\max_{s \in S} \sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s, a),
\]
subject to the constraint \( \sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s, a) = 0 \). By Assumption 1, the solution to the above maximization can offer the buyer a higher payoff than her outside option for all types \( \theta \in \Theta \).

A set of full-information optimal contracts is on the Pareto frontier for all \( \theta \) and offers lenders zero expected utility. As a result, for any deviating contract that a lender might be willing to offer, if the borrower correctly inferred the lenders’ true type, then the borrower would weakly prefer the full-information optimal contract being offered. The \( D1 \) refinement in our model allows the borrower to make this inference, and the presence of a competing lender allows the borrower to choose the equilibrium full-information optimal contract instead of the deviating contract. The following proposition summarizes this discussion.

**PROPOSITION 2:** Under Assumption 1, the symmetric pure-strategy equilibrium \( s(\theta) = \bar{s}(\theta) \) exists.

**Proof:** See Appendix B.

This proposition describes a “best” symmetric pure-strategy equilibrium, in which a full-information optimal contract is offered.\(^{18}\) Our main results describe the conditions under which another type of symmetric pure-strategy equilibrium exists. This alternative equilibrium is notable because it uses a noncontingent contract, is a pooling equilibrium, and is Pareto-inferior to the “best” equilibrium, from an ex-ante perspective.

We say that a contract is “noncontingent” if \( s_z = s_{z'} \) for all \( z, z' \in Z \); that is, the contract does not make use of the index. We consider the existence of a noncontingent contract pooling equilibrium, in

\(^{18}\)Note that there is a tension between the existence of the best equilibrium, which uses indexed (contingent on \( z \in Z \)) contracts, and the assumption that the lender’s indirect utility function depends only on the external state and not directly on the index. For example, if the lender made a number of other indexed loans, then the lender’s marginal utility might be a function of both \( a \in A \) and \( z \in Z \). Because the focus of our analysis is the existence of an equilibrium without indexation, we do not discuss this issue in more detail.
which, for all $\theta \in \Theta$ with $\mu_0(\theta) > 0$,

$$s_z(\theta) = s^* \in \arg\max_{s \in S, a \in A} \sum_{a \in A} p(a) \phi_B(s, a),$$

subject to $\sum_{a \in A} p(a) \phi_L(s, a) = K$.

By Assumption 1, this contract can offer the buyer a higher payoff than her outside option for all types $\theta \in \Theta$. Note also that this equilibrium is at least weakly Pareto-inferior to the “best” equilibrium and is strictly inferior if the noncontingent contract $s^*$ is suboptimal for any type $\theta$ under full information.

**V. Risk-Sharing Failure in Equilibrium**

In this section, we provide sufficient conditions for the existence of a “noncontingent” equilibrium. This equilibrium will exist despite its ex-ante Pareto-inferiority to the “best” equilibrium discussed above.

Our second assumption states that the value of a larger promise to the lender is higher in bad external states than in good external states. In other words, the lender is risk-averse with respect to promises.

**ASSUMPTION 2:** For all $d', d \in D$ with $d' > d$, $\phi_L(s_{d'}, a) - \phi_L(s_d, a)$ is weakly decreasing on $a \in A$.

From the lender’s perspective, it is preferable to receive larger promises in worse states. In other words, $\phi_L(s_d, a)$ is submodular in $(d, a)$. This is what we mean by the idea that “lenders are risk-averse with respect to promises.”

Our third assumption is defined using the variable $\lambda^*$, which is the Pareto weight associated with the noncontingent contract $s^*$:

$$s^* \in \arg\max_{s \in S, a \in A} \sum_{a \in A} p(a) U(s, a; \lambda^*),$$

where

$$U(s, a; \lambda) = \phi_B(s, a) + \lambda \phi_L(s, a)$$

is the social welfare function. The Pareto weight $\lambda^* > 0$ is also the multiplier on the constraint in equation (15), and hence $s^*$ causes the lender to receive zero expected utility.

Our assumption requires that the marginal social value of a promise to a lender is lower in bad external states than in good external states. In other words, it is efficient to the lender to insure the borrower.

**ASSUMPTION 3:** For all $d, d' \in D$ with $d' > d$, $U(s_{d'}, a; \lambda^*) - U(s_d, a; \lambda^*)$ is weakly increasing on $a \in A$.

The social welfare function is supermodular in $(d, a)$, which implies among other things that the
borrower’s indirect utility function, $\phi_B(s_d, a)$, is supermodular in $(d, a)$. That is, this assumption implicitly embeds the assumption that the borrower is also “risk-averse with respect to promises.”

These two assumptions can be understood as consisting of several claims. The first claim is that the “marginal benefit of debt” to the lender, $\phi_L(s_d, a) - \phi_L(s_d', a)$, is monotone in the aggregate state, regardless of the levels of debt involved. The first part of this claim can be thought of as defining the order on the aggregate states—up to this point, nothing has depended on that order. The second part (“regardless of the level of debt”) is the key point. The second claim is that the “marginal cost of debt” to the borrower, $\phi_B(s_d, a) - \phi_B(s_d', a)$, is monotone and increases in the same direction as the marginal benefit of debt to the lender. In other words, states in which the lender would really like larger promises are also states in which the borrower would really prefer not to make larger promises. The third claim is that the borrower is “more risk-averse” than the lender in this sense. That is, in states in which the lender would really like a large promise and the borrower would really prefer a small promise, the latter effect dominates, and under the Pareto weight $\lambda^*$, it is more efficient to have smaller promises when both “marginal cost” and “marginal benefit” are high. In other words, the optimal contract would involve the lender insuring the borrower, and because their preferences are aligned, this is costly for the lender.

As suggested by this description, our results do not really depend on the ordering over the external states, that is, the proof of Proposition 3 that follows would hold almost unchanged if we instead impose the two assumptions that $\phi_L(s_d, a)$ was supermodular and $U(s_d, a; \lambda^*)$ was submodular.

Our last assumption requires that the set of possible types (the support of the prior $\mu_0$) be sufficiently rich, in the sense that there is always a “less interrelated” type. There are a variety of ways to define “less interrelated” in the context of joint probability distributions with identical marginal distributions. For two variables (i.e., $a \in A$ and $z \in Z$), many of these orders are equivalent (Meyer and Strulovici (2012)). One intuitive way to measure interrelatedness is the greater weak association relation defined by Meyer and Strulovici (2012). A definition, in our context, is as follows.

**DEFINITION 1:** A type $\theta \in \Theta$ has greater weak association than a type $\theta' \in \Theta$, $\theta \succeq_{GWA} \theta'$, if, for all nondecreasing functions $h : A \to \mathbb{R}$ and $g : Z \to \mathbb{R}$,

$$\text{Cov}^\theta(h(a), g(z)) \geq \text{Cov}^{\theta'}(h(a), g(z)).$$

In other words, the type $\theta'$ has less correlation than the type $\theta$, regardless of how the external states $a$ and index values $z$ are mapped to real numbers. One consequence of $\theta \succeq_{GWA} \theta'$ is that (again from Meyer and Strulovici (2012))

$$\sum_{a \in A, z \in Z} (\theta(a, z) - \theta'(a, z)) f(a, z) \geq 0$$

for all supermodular functions $f$. We can also relate greater weak association to the more familiar notion of first-order stochastic dominance. If $\theta \succeq_{GWA} \theta'$, then for all $z' \in Z$, the conditional distribution $\theta(a|z \geq z')$ first-order stochastically dominates $\theta'(a|z \geq z')$, and $\theta'(a|z < z')$ first-order stochasti-
cally dominates \( \theta(a|z < z') \). That is, if \( \theta \succeq_{GW A} \theta' \), then higher values of the index are more strongly associated with better external states under \( \theta \) than under \( \theta' \).

We assume that every type \( \theta \) in the support of \( \mu_0 \) has greater weak association than the uninformative type \( \theta_0 \) (i.e., all \( \theta \in \Theta \) are weakly associated). The key part of our assumption is that “it can always be worse.” In other words, for any type \( \theta \) that is possible (\( \mu_0(\theta) > 0 \)), every type that is less interrelated is also possible, including the uninformative type.

**ASSUMPTION 4**: For all \( \theta \in \Theta \) in the support of \( \mu_0 \), \( \theta \succeq_{GW A} \theta_0 \), if \( \mu(\theta) > 0 \), then \( \mu(\theta') > 0 \) for all \( \theta' \in \Theta \) such that \( \theta \succeq_{GW A} \theta' \succeq_{GW A} \theta_0 \).

This assumption ensures that for any type, there is a rich set of less informative types, which limits the lender’s ability to simultaneously signal her type and capture risk-sharing benefits.

Before proving our main result, we note that we have made Assumptions 2 and 3 weak so that they are easy to satisfy. In this spirit, we have also imposed relatively little structure on the indirect utility functions \( \phi_L \) and \( \phi_B \). Consequently, although it is guaranteed that the full-information optimal contract is weakly better than the noncontingent contract, we have not assumed enough to show that it is strictly better. We now provide the following lemma to show that stronger versions of our assumptions are sufficient but not necessary to ensure that the full-information optimal contract is strictly better.

Recall that both the full-information optimal contract \( \bar{s}(\theta) \) and the non-contingent optimal contract \( s^* \) are designed to ensure that the lender earns zero expected utility.

**LEMMA 2**: Let \( d(s^*) \) denote the value of \( d \in D \) associated with the non-contingent optimal contract \( s^* \), and suppose it is in the interior of \( D \). If \( \phi_B(s_d, a) \) and \( \phi_L(s_d, a) \) are both differentiable with respect to \( d \) at \( d(s^*) \) and

\[
\frac{\partial}{\partial d} U(s_d, a; \lambda^*)|_{d = d^*(s)}
\]

is strictly increasing on \( a \in A \), then for all types \( \theta \in \Theta \) such that \( \theta \succeq_{GW A} \theta_0 \), except \( \theta_0 \) itself, the full-information optimal contract is strictly Pareto-superior to the non-contingent contract,

\[
\sum_{a \in A, z \in Z} \theta(a, z) \phi_B(\bar{s}_z(\theta), a) > \sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s^*, a).
\]

**Proof**: See Appendix B.

The stronger assumptions of this lemma rule out some technical issues such as the possibility that full-information optimal contract is noncontingent due to some kind of boundary or discontinuity, or that there are no risk-sharing benefits. In the examples we have constructed, these issues arise only in pathological cases and the full-information optimal contract is indeed better than the noncontingent contract for almost all \( \theta \) in the support of \( \mu_0 \). This leads to the following “puzzle” in our model: can the noncontingent equilibrium exist even though the best equilibrium is ex-ante strictly Pareto-superior? Our main result answers this question in the affirmative.
PROPOSITION 3: Under Assumptions 1, 2, 3, and 4, there exists a symmetric pure-strategy equilibrium in which $s(\theta) = s^*$. 

Proof. See Appendix B. The proof relies on results from Meyer and Strulovici (2015).

This proposition establishes that the assumptions given above are sufficient for the existence of a noncontingent equilibrium. Intuitively, if it is not efficient for the borrower to insure the lender, then the deviations necessary to separate from the uninformative type are never welfare-improving. Our conditions are designed to ensure that this is the case.

Having presented the main result, we briefly comment on the importance of each of our assumptions. Assumption 1 ensures that both the full-information optimal contract and the noncontingent optimal contract are feasible from a participation constraint perspective. However, because the full-information optimal contract can strictly Pareto-dominate the noncontingent contract for many types, under an alternative assumption it is possible to have the full-information optimal contract feasible for some types, while the noncontingent contract is infeasible. In this case, there could not be a noncontingent equilibrium. However, the proof of Proposition 3 could be adapted to prove that a “no trade” equilibrium exists in this case, despite the possibility of gains from trade for some types.

Assumption 2, lender risk-aversion with respect to promises, is essential to the result. If the lender were risk-seeking with respect to promises while the borrower remained risk-averse with respect to promises, then lenders with a more accurate index could separate from the uninformative type by paying higher prices to provide insurance. Assumption 3, which implies that it is efficient for the lender to insure the borrower and not vice versa, is essential for similar reasons. If it were instead optimal for the lender to purchase insurance from the borrower, then a lender with a more accurate index could separate from the uninformative type by paying a high price for insurance.\(^{19}\)

Assumption 4 is essential to rule out nonmonotone (in $z$) security designs. If the security were required to be monotone in $z$ (but allowed to be either increasing or decreasing), then the possibility of the uninformative type $\theta_0$ would be sufficient to generate the noncontingent equilibrium. Using nonmonotone securities potentially allows a lender to purchase insurance over some subset of $Z$ at a high price, separating from the uninformative type, while providing insurance over another subset of $Z$. This would perhaps generate enough gains from trade to make such a deviation worthwhile. The richness of the type space allows the borrower to be suspicious of such an offer, thinking that the index is likely to work well over the subset of $Z$ for which the lender is buying insurance but poorly over the subset for which the borrower is buying insurance.

Assumptions 1, 2, and 3 are properties of the indirect utility functions $\varphi_B$ and $\varphi_L$ and the required funds $K$. As a result, they can be checked in the context of a specific model, such as the mortgage model presented previously. For another example, which uses a CSV model, see Appendix A.

Having presented our main result, we next turn to model variations and extensions.

\(^{19}\)We do not mean to imply that our assumptions are necessary; they are only sufficient, and we speculate that our result could be proven under modified versions of these assumptions.
VI. Variations and Extensions

In this section, we discuss modifications and extensions to the model. We begin by discussing a model with positive profits for lenders, which nevertheless retains the competition between lenders. In this case, our results go through essentially unchanged. We then discuss what would happen under a single, monopoly lender. We find that no full-information optimal contracts equilibrium arises under a monopoly lender but a noncontingent equilibrium continues to hold. Finally, we discuss how our results can be extended to settings in which there is adverse selection about the marginal distribution of the index \((q)\), or about the distribution of the external states but not about the index quality.

A. Profitable Lending

In this extension, we describe a model in which lenders make positive profits, that is, receive expected utility greater than their outside option, in equilibrium but nevertheless face competition. We introduce profits into the economy by assuming that each lender faces a convex cost in the number of loans she makes and that there is a unit mass of borrowers.\(^{20}\) Let \(Q_l\) be the number of loans made by lender \(l\). Suppose that a lender of type \(\theta\) that makes \(Q\) loans using contract \(s\) receives utility

\[
\Pi(s, Q, \theta) = Q \left\{ \sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s, z, a) \right\} - C(Q),
\]

(17)

where \(C(Q)\) is a convex, twice-differentiable function with \(C'(|L|^{-1}) = 0\).

With this quasi-linear functional form and the normalization that \(C'(|L|^{-1}) = 0\), a lender that considers a deviation in which the lender offers a single, marginal borrower a different contract faces a problem that is identical to the one considered in our general model. In this case, the \(D_1\) refinement is the same as in our main analysis, because (in equilibrium) the marginal profit of each lender is zero.

However, if the lender contemplates a deviation in which she offers a deviating contract to all borrowers, then substantial profits could be at stake because the average profits of lenders are positive. In this case, the \(D_1\) refinement requires that the borrower place her beliefs on the lender type that would break even under the smallest amount of demand for the deviating contract. This is equivalent to saying that the borrower must believe that the lender is of a type for whom the difference between the marginal profit of the deviating contract and the marginal profit of the equilibrium contract is maximal.

Perhaps surprisingly, our noncontingent equilibrium exists under the same conditions in this model. The intuition comes from the proof description in Section V. When a lender with a “good” index offers a contract that insures the borrower, the lender requires a higher expected value of repayments to be indifferent between the deviating contract and the noncontingent contract. However, a lender with an irrelevant index could offer the same deviating contract at a profit and thus (in the

\(^{20}\)Introducing profits in this way is an old idea. See Tirole (1988).
case of profitable lending) the borrower must believe that the lender is of this type or of a type that is even worse from the perspective of the borrower.

B. Monopoly Lending

In this extension, we consider the type of equilibrium that can exist when the lender has monopoly power. Specifically, we assume that a single lender can make a take-it-or-leave-it offer to the borrower and if the borrower rejects this offer, then she receives her outside option. Neither the full-information optimal contract nor the noncontingent contract defined in Section IV are equilibria because both offer positive surplus to the borrower and zero surplus to the lender.

To study the monopoly case, we parameterize both the full-information optimal contract and the noncontingent contract by the required investment. Suppose that there exists a $\bar{K} > K$ such that the full-information optimal contract, $\bar{s}(\theta, \bar{K})$, results in a payoff for the borrower equal to her outside option. Likewise, suppose that there exists a $K^* > K$ such that the noncontingent contract, $s^*(K^*)$, also results in a payoff for the borrower equal to her outside option. In this subsection we ask whether there exist equilibria with contracts $\bar{s}(\theta, \bar{K})$ and $s^*(K^*)$. We continue to impose the $D1$ refinement on off-equilibrium contract offers.

The answer is no for the full-information contract and yes for the noncontingent contract. The existence of the noncontingent contracts equilibria follows from the proof of Proposition 3; nothing in that proof depends on a specific value of $K$. The only effect of competition is to allow the borrower to choose a contract from another lender. Although the type $\theta$ is common to all lenders, because the noncontingent contract’s payoff for the borrower does not depend on $\theta$, the borrower’s inference about $\theta$ does not change the appeal of the noncontingent contract. It is as if the borrower had a fixed outside option instead, which is what is assumed in the monopoly case.

However, for the full-information contract, competition is essential. For the uninformative type ($\theta_0$), the full-information contract is identical to a noncontingent contract. For many other types (by Lemma 2), the full-information contract is contingent and, due to lender risk-aversion over promises (Assumption 2), offers a higher payoff to the uninformative type than the noncontingent contract. As a result, the uninformative type is tempted to deviate. When there are other lenders, the borrower can use their offers to determine the common type and avoid being “tricked” by this deviation. However, this is not possible with a monopoly lender and as a result there is no full-information contract equilibrium. In summary, competition is necessary for the existence of the best equilibrium but the noncontingent equilibrium always exists.

Note that this result offers a hysteresis-based explanation for why we might expect the noncontingent equilibrium to occur despite the presence of competition. If, at the beginning of the market, there was only one lender, then the noncontingent contract would be used. This might anchor the borrower’s expectations, so that as other competing lenders entered, the noncontingent contract would continue to be employed. The entry of lenders would still benefit the borrower, due to better pricing (the difference between $K^*$ and $K$ described above), but would not achieve the full benefit of allowing for contingent contracts.
C. Adverse Selection of Marginal Distributions

Throughout this paper we assume that the set $\Theta$ contains only joint distributions of the external state and the index with marginal distributions $p(a)$ and $q(z)$. Suppose that we relax this assumption and require only that the marginal distribution over external states, $p(a)$, be the same for all types. Under this assumption, there is no adverse selection about the true external state, but rather only about the index (as in the main part of the paper). Intuitively, adding additional dimensions of adverse selection cannot improve the situation and should only reinforce the noncontingent contracts equilibrium.

Formally, let $q(z; \theta)$ denote the marginal distribution of the index associated with type $\theta$, and let $\Theta(q)$ be the set of all joint distributions with marginals $q(z)$ and $p(a)$, satisfying the monotone likelihood ratio property for the conditional distribution of $a$ given $z$. Let $Q$ be the set of all marginal distributions for the index, and let $\Theta$ be the union of all $\Theta(q)$ for each $q \in Q$. We modify our Assumption 4 (rich type space) so that it applies to each $\Theta(q)$ such that $\mu(\theta) > 0$ for some $\theta \in \Theta(q)$. In other words, there is a rich type space and an opportunity for risk-sharing for each possible marginal distribution of the index. Under this condition, the proof of Proposition 3 is essentially unchanged and the result holds.

D. Adverse Selection of External States

In this extension, we modify the model of the main text to consider the case in which the index is known to be perfect but there is adverse selection about the marginal distribution of the aggregate state. This sort of adverse selection is closer to the problems studied in the literature (e.g., Spier (1992), Asriyan (2015)). We build on the notation used in the previous extension. We assume that the set $Z$ is identical to the set $A$, and that each $\Theta(q)$ is a singleton, containing only the joint distribution

$$\theta(a, z) = \delta(a, z) q(z),$$

where $\delta(a, z)$ is equal to one if $a = z$ and to zero otherwise. Adverse selection occurs because, in this context, there are types in $\Theta$ with different values of $q(z; \theta)$.

In this setting, the “rich type space” assumption (Assumption 4) is irrelevant. We continue to impose our feasibility assumption (Assumption 1) for each $\Theta(q)$. Our result in this section does not depend on detailed assumptions about risk-sharing (like Assumptions 2 and 3), and thus we do not discuss how to adapt them to this setting. We assume instead the result of Lemma 2, namely that the full-information optimal contract is contingent for all $\theta$ in the support of $\mu_0$ and generates strictly higher payoffs than any noncontingent contract. We discuss how to weaken these assumptions below. We also assume that both $\phi_L$ and $\phi_B$ are continuous in $d$.

For technical reasons, we assume that the support of the prior beliefs $\mu_0(\cdot)$ is a closed set, which is not required in the main analysis. We also assume that all types $\theta \in \Theta$ for which $\mu_0(\theta) > 0$ are associated with marginal distributions that have full support. In other words, $q(z; \theta) > 0$ for all $z \in Z$ and $\theta \in \Theta$ such that $\mu_0(\theta) > 0$. This generalizes the full-support assumption of the main analysis.
Define the mapping $\Theta^*(\theta)$ as a set-valued function

$$\Theta^*(\theta) = \{ \theta' \in \arg \max_{\theta' \in \Theta; \mu_0(\theta') > 0} \sum_{a \in A, z \in Z} \theta'(a, z) \phi_L(\bar{s}_z(\theta), a) \},$$

(19)

where $\bar{s}_z(\theta)$ is the full-information optimal contract associated with the type $\theta$. The set $\Theta^*(\theta)$ is the set of types in the support of $\mu_0(\cdot)$ that would earn the highest payoff from offering the security $\bar{s}_z(\theta)$.

The following lemma (which is based on standard fixed-point arguments) states that there is a fixed point to this mapping.

**Lemma 3:** There exists a $\theta^*$ such that $\theta^* \in \Theta^*(\theta^*)$. For any such $\theta^*$, for all $\theta' \in \Theta^*(\theta^*)$, $\bar{s}(\theta') = \bar{s}(\theta^*)$.

**Proof.** See Appendix B.

This type, $\theta^*$, can essentially “prove itself” by offering its full-information optimal contract. When a lender of type $\theta^*$ offers the contract $\bar{s}(\theta^*)$, it breaks even. All other types either recover less than the initial investment $K$, or also break even and have an identical full-information optimal contract. Hence, under the $D_1$ refinement, the borrower must believe that she is being offered a full-information optimal contract.

By assumption, every full-information optimal contract is not equal to a noncontingent contract. By the Pareto-optimality of the full-information optimal contract, the borrower must be willing to accept this contract. Therefore, there cannot be an equilibrium in which a noncontingent contract is employed.$^{21}$

What makes this setting different than that studied in the main analysis? The key here is that there is no type for which the full-information optimal contract is equal to the noncontingent contract. When there is adverse selection about the distribution of external states, this makes sense; the only way a noncontingent contract could be optimal is if some type of lender knew with certainty what the ex-post “fair” level of debt is. In contrast, in the case emphasized in the main analysis, a noncontingent contract can be optimal as long as it is possible that the index is irrelevant; perfect foresight about the external state is not required for a noncontingent contracts equilibrium.

This result can be viewed as pointing to the necessity of something like Assumption 4 in the main analysis. If the borrower knew that the index is at least somewhat relevant, then the type with the least relevant index could “prove himself” and eliminate the noncontingent equilibrium. In this case, an equilibrium with a minimal (and ex-ante suboptimal) level of indexing would exist. Non-contingency could be restored in this case by introducing a fixed cost of using the index in addition to asymmetric information, following Spier (1992). In this case, a noncontingent equilibrium would exist as long as the “worst” type was sufficiently bad, relative to the fixed cost.

Note also that, consistent with the Hirshleifer effect (Hirshleifer (1971)), with adverse selection on external states, the risk-sharing will generally be reduced relative to the case in which the lenders

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$^{21}$This argument does not really depend on every full-information optimal contract being contingent: rather, only that the type that can prove itself has a contingent full-information optimal contract. To formalize this argument, we would need to use continuity to show that a lender can deviate to a contract close to the full-information contract but with a tiny profit and that the borrower cannot reject such a contract.
shared the borrower's prior. That is, although one type can “prove itself,” most types will not be able to do so and the equilibrium will likely involve less risk-sharing than if the lenders were uninformed and shared the borrower's prior. Our point is that under our assumptions this reduction in risk sharing never enough to generate a noncontingent contracts equilibrium. In contrast, adverse selection of the relationship of the index and the external state, as studied in the main part of this paper, can generate a noncontingent contracts equilibrium.

VII. Conclusion

In this paper we introduce a theory to explain the widespread lack of indexation observed in contracts. Intuitively, when a borrower is offered a contract that includes insurance, she may be concerned that the insurance is not relevant for the risks she faces. Under the conditions described in our model, this effect can be strong enough to lead the borrower to reject the offer and instead choose a contract without insurance from a different lender. As a result, equilibria that feature little or no risk-sharing can arise, even though they are ex-ante Pareto-dominated by equilibria that feature full risk-sharing.
REFERENCES


Segal, Ilya, and Michael D Whinston, 2002, The Mirrlees approach to mechanism design with renegotiation (with applications to hold-up and risk sharing), *Econometrica* 70, 1–45.


Sinai, Todd, and Nicholas S. Souleles, 2005, Owner-occupied housing as a hedge against rent risk., *Quarterly Journal of Economics* 120.


**Appendix A. Costly State Verification Example**

In this appendix, we provide a numerical example under which a version of the CSV model (e.g. Townsend (1979) and Gale and Hellwig (1985)) satisfies the conditions of our main theorem, particularly Assumptions 2 and 3. We introduce some functional forms to apply the results of our
general model as simply as possible. The key modifications to the standard CSV model are the introduction of an external state, borrower risk-aversion, and borrower nonverifiable income. Better external states lead to a better (in a monotone-likelihood-ratio property sense) distribution of both verifiable and nonverifiable income. The presence of nonverifiable income, when combined with risk-aversion, implies that the borrower has low expected marginal utility in good external states, as in our mortgage example (Section II).

We now describe the specific modifications that we make to the standard CSV model. The idiosyncratic state (as opposed to external state) for the borrower is a triple \((x, y, y')\), where \(x\) is the borrower’s nonverifiable income, \(y\) is the borrower’s verifiable income, and \(y'\) is the borrower’s report of her verifiable income, all of which are weakly positive reals. The conditional contract \(s(y, y')\) can depend on both the true verifiable income and the report.\(^{22}\) The borrower’s utility is given by

\[
u(x + y - s(y, y')),\]

where \(\nu(\cdot)\) is the borrower’s strictly increasing, twice-differentiable concave utility function. Our numerical example uses CARA utility for the borrower. We do not analyze the borrower’s participation constraint but rather make the assumption that the project is sufficiently valuable from the borrower’s perspective to always be enough to ensure that the participation constraint is satisfied.

We assume that there are two equally likely external states, \(A = \{9_{10}, 11_{10}\}\), and two equally likely index values, \(Z = \{z_l, z_h\}\), as in our two examples in the main text. We further assume that the quality of the index (\(\rho\), see Section II) is at most \(\frac{1}{8}\).\(^{23}\)

Let \(f(y|a)\) denote the distribution of \(y\) given \(a\), and suppose that it has the functional form

\[
f(y|a) = q(y)\exp(a \ln(y) - \psi(a)),\]

where \(q(y) = f(y|0)\) is a measure on the positive reals. In other words, the distributions \(f(\cdot|a)\) are an exponential family whose sufficient statistic is the expected log verifiable income. The function \(\psi(a)\) ensures that each \(f(y|a)\) integrates to one. In our numerical calculations, we assume that \(q(y)\) is Gamma-distributed with shape parameter four and scale parameter \(\frac{1}{2}\). As a result, the distributions \(f(y|a = 9_{10})\) and \(f(y|a = 11_{10})\) are also Gamma-distributed, with the same scale parameter and shape parameters slightly below and above five, respectively. We use the Gamma distribution because it generates tractable expressions for the integrals that define marginal utility and because it has a “hump” shape.

We suppose (for tractability) that the nonverifiable income \(x\) is equal to

\[
x = \chi a y,\]

where \(\chi\) is a positive constant. In our numerical exercise, we set \(\chi = 1\), which means that roughly half

\(^{22}\)It is without loss of generality to assume that the contract does not depend on a report of the borrower’s non-verifiable income because the borrower would report the level of nonverifiable income level that minimizes her repayment.

\(^{23}\)The maximum value of \(\rho\) matters only when determining the largest debt level that must be considered.
of the borrower’s income is nonverifiable. We have in mind, for example, future labor income. Under our assumptions, the expected total pledgeable and nonpledgeable income is five and the standard deviation of total income is roughly half this value. The expected total income is about 15% higher in the good external state than the bad external state. We set the cost of the project, $K$, to one.

One consequence of our assumptions is that the borrower, observing $x$ and $y$, can infer the true external state $a$. This implication is by no means necessary—we could add noise to the value of $x$ and prevent the borrower from inferring $a$, at the cost of having a more complicated example.

We now turn to the lender. The lender is risk-neutral within each external state (i.e., with respect to the borrower’s idiosyncratic state), but risk-averse with respect to the external state. Let $M(a)$ denote the lender’s marginal utility given the external state. We use the notation $M(a)$ to emphasize that this can be interpreted as the lender’s SDF. We use the functional form $M(a) = Ma^{-\frac{1}{2}}$, setting $\bar{M}$ so that the expected value of $M(a)$ is equal to one. By interpreting $M(a)$ as an SDF, this sets the risk-free rate to zero.

If the conditional (on $z \in \mathcal{Z}$) contract depends on the true value, then for a given value of the report, the lender pays a verification cost. Let $c(y';s) = \bar{c} > 0$ if there exists a $y_1, y_2$ such that the conditional contract $s$ offers different payments for $(y_1, y')$ and $(y_2, y')$, and let $c(y';s) = 0$ otherwise. In our numerical calculation, we use $\bar{c} = 10\%$, recalling that we normalize the project size to one.

We next describe the general forms of the indirect utility functions. Let $\omega(y'|y, x)$ denote a (possibly mixed) reporting strategy by the borrower, and let $\omega^*(y'|y, x)$ be the optimal reporting strategy. The indirect utility functions are

$$\phi_B(s, a) = \max_{\omega(y'|y, x) \in \Omega(s)} \int_{0}^{\infty} \int_{0}^{\infty} u(\chi ay + y - s(y, y')) \omega(y'|y, \chi ay)f(y|a)d'y'dy,$$

$$\phi_L(s, a) = M(a) \int_{0}^{\infty} \int_{0}^{\infty} (s(y, y') - c(y';s) - K) \omega^*(y'|y, \chi ay)f(y|a)d'y'dy,$$

where $\Omega(s)$ denotes the constraints in the reporting strategy, which we describe next. We impose limited liability, which means that for all reports $y'$, either $s(y_1, y') = s(y_2, y')$ for all $y_1, y_2$ and $0 \leq s(y', y') \leq y'$ (the nonverification case) or $0 \leq s(y, y') \leq y$ for all $y$ (the verification case). We restrict the reporting strategies $\omega(y'|y, x)$ to place support only on $y'$ for which the reports are feasible, which means that if $s(y_1, y') = s(y_2, y')$ for all $y_1, y_2$, then $\omega(y'|y, x) = 0$ if $y < s(y', y')$. In other words, for reports that do not trigger verification, the borrower must have the funds to repay the loan.

Although this model is slightly different from that in Townsend (1979) and Gale and Hellwig (1985), the arguments for the optimality of a debt contract are essentially unchanged. By fixing some distribution over external states $p(a)$ and taking the expected value of the indirect utility functions, it is immediately apparent that the model is exactly that of Townsend (1979) except that it has nonverifiable income. However, the argument for the optimality of debt depends only on nonsatiation and on the condition that utility at zero verifiable income net of debt repayments not be infinite. Therefore, with either $u'(0) > -\infty$ or $x > 0$ with probability one, debt will be optimal for all $p(a)$. It follows that debt is “ex-post optimal” in the sense assumed in the main text, and thus we restrict
attention to contracts that are (possibly indexed on $z \in Z$) debt contracts.

The set of feasible debt levels is $D = [0, \bar{d}]$. The level of $\bar{d}$ is determined by the smallest promise such that if the promise for the other value of $z \in Z$ is zero, then all lender types at least break even. Any promise larger than this will necessarily generate profits for all lender types and can therefore be rejected by the borrower. The value of $\bar{d}$ is determined by the $\phi_L$ function and our assumption on the set of possible index qualities; we omit the details for brevity.

Specializing the indirect utility functions to a debt contract induces truthful reporting,

$$
\phi_B(s,d,a) = \int_d^\infty u((1 + \chi a)y - d)f(y|a)dy
+ \int_0^d u(\chi ay)f(y|a)dy,
$$

and

$$
\phi_L(s,d,a) = \int_d^\infty (d - K)f(y|a)dy
+ \int_0^d (y - K - \bar{c})f(y|a)dy.
$$

We note that these indirect utility functions satisfy the assumptions that we impose. In particular, they are both differentiable (and hence continuous) in $d$ and the lender’s indirect utility function is negative when the level of debt is zero. The derivative of the borrower’s indirect utility function with respect to the level of debt is

$$
\phi_{B,d}(s,d,a) = -\int_d^\infty u'((1 + \chi a)y - d)f(y|a)dy,
$$

and hence is strictly negative, which satisfies our monotonicity requirement. The derivative of the lender’s indirect utility function is

$$
\phi_{L,d}(s,d,a) = -M(a)f(d|a)\bar{c} + M(a)\int_d^\infty f(y|a)dy.
$$

As in the mortgage example of Section II, several key forces determine whether Assumptions 2 and 3 are satisfied. Loosely speaking, lender risk-aversion (decreasing $M(a)$) must outweigh the increasing likelihood of not being repaid in bad states, while borrower risk-aversion with respect to promises must sufficiently dominate lender risk-aversion with respect to promises.

We now turn to our numerical analysis. The Table A.I summarizes our functional forms and parameter assumptions. Under these functional forms and assumptions, we verify using Mathematica that Assumptions 2 and 3 are satisfied for all $a \in \{\frac{9}{10}, \frac{11}{10}\}$ and $d \in [0, \bar{d}]$. 

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Table A.I  
Parameters of CSV Example

<table>
<thead>
<tr>
<th>Function/Parameter</th>
<th>Functional Form</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Utility $u(\cdot)$</td>
<td>CARA</td>
<td>1</td>
</tr>
<tr>
<td>Lender Marginal Utility $M(a) = \bar{M}a^{-\frac{1}{2}}$</td>
<td>$\bar{M} = [\frac{1}{2}(\frac{9}{10})^{-\frac{1}{2}} + \frac{1}{2}(\frac{11}{10})^{-\frac{1}{2}}]^{-1}$</td>
<td></td>
</tr>
<tr>
<td>PDF $q(y)$</td>
<td>Gamma($\kappa, \theta$)</td>
<td>$\kappa = 4, \theta = \frac{1}{2}$</td>
</tr>
<tr>
<td>Verification cost $\bar{c}$</td>
<td>$\bar{c} = \frac{1}{10}$</td>
<td></td>
</tr>
<tr>
<td>N.V. Income Function $\mu(y, a)$</td>
<td>$\mu(y, a) = \chi ya$</td>
<td>$\chi = 1$</td>
</tr>
<tr>
<td>Required Funds $K$</td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>External States $A$</td>
<td></td>
<td>${\frac{9}{10}, \frac{11}{10}}$</td>
</tr>
<tr>
<td>Probabilities $p(a)$</td>
<td></td>
<td>$(\frac{2}{2}, \frac{2}{2})$</td>
</tr>
</tbody>
</table>

Appendix B.  Proofs

Proof of Proposition 1. We prove Proposition 1 by first demonstrating that the model described in Section II satisfies the assumptions of our general model described in Sections III, IV, and V, and then relying on our main result in Proposition 3. Consequently, this proof uses notation and refers to assumptions that are introduced in the text after Proposition 1. We encourage readers to read Sections III, IV, and V of the main text before examining this proof.

To prove Proposition 1 we show that Assumptions 1, 2, 3, and 4 of the main text are satisfied. We begin by showing that Assumption 2, lender submodularity, is satisfied. The lender’s indirect utility function $\phi_L(s_d, a)$, is submodular, in a differentiable context, if

$$\phi_{L,d}(s_d, a) = \pi(a)u'_{L}(y_L(a))$$

is decreasing on $a \in A$. With two states and CRRA, this condition is equivalent to

$$y_L(a_{good})^{-\gamma_L} \pi(a_{good}) \leq y_L(a_{bad})^{-\gamma_L} \pi(a_{bad}).$$

Therefore $\phi_L(s_d, a)$ is submodular by equation (1).

We next show that the social welfare function is supermodular (Assumption 3). This assumption requires that

$$U_d(s_d, a; \lambda^*)$$

be increasing in $a$. We can write

$$U_d(s_d, a; \lambda^*) = \phi_{B,d}(s_d, a) + \lambda^* \phi_{L,d}(s_d, a).$$
The definition of \( \lambda^* \) is (by equation (16))

\[
\sum_{a \in A} p(a)U_d(s^*, a; \lambda^*) = 0,
\]

noting that an interior solution in equation (16) is guaranteed for \( K \), and hence \( d^* \), sufficiently small. It follows that

\[
\lambda^* = \frac{\phi_{B,d}(s^*, a_g) + \phi_{B,d}(s^*, a_b)}{\phi_{L,d}(s^*, a_g) + \phi_{L,d}(s^*, a_b)}.
\]

The denominator is positive because \( \lambda^* > 0 \). We now observe that the social welfare function is supermodular if and only if

\[
U_d(s_d, a; \lambda^*)(\phi_{L,d}(s^*, a_g) + \phi_{L,d}(s^*, a_b))
\]
is increasing in \( a \). Using the solution for \( \lambda^* \), this condition is equivalent to

\[
\phi_{B,d}(s_d, a_g)[\phi_{L,d}(s^*, a_g) + \phi_{L,d}(s^*, a_b)] - \phi_{L,d}(s_d, a_g)[\phi_{B,d}(s^*, a_g) + \phi_{B,d}(s^*, a_b)] \geq

\phi_{B,d}(s_d, a_b)[\phi_{L,d}(s^*, a_g) + \phi_{L,d}(s^*, a_b)] - \phi_{L,d}(s_d, a_b)[\phi_{B,d}(s^*, a_g) + \phi_{B,d}(s^*, a_b)].
\]

We can rewrite this as

\[
[\phi_{B,d}(s_d, a_g) - \phi_{B,d}(s_d, a_b)][\phi_{L,d}(s^*, a_g) + \phi_{L,d}(s^*, a_b)] \geq

[\phi_{L,d}(s_d, a_g) - \phi_{L,d}(s_d, a_b)][\phi_{B,d}(s^*, a_g) + \phi_{B,d}(s^*, a_b)].
\]

(B.1)

Observe that, for all \( d, d' \in D \),

\[
\phi_{L,d}(s_d, a) = \phi_{L,d}(s_{d'}, a).
\]

We have

\[
\phi_{B,d}(s_d, a) = -\pi(a)u_B'(y_B(a) + x_h - d)
\]

and therefore

\[
\ln(-\frac{\phi_{B,d}(s_d, a_g)}{-\phi_{B,d}(s_d, a_b)}) = \ln(\frac{\pi(a_g)}{\pi(a_b)}) = \gamma_B \ln(\frac{y_B(a_g) + x_h - d}{y_B(a_b) + x_h - d}).
\]

It follows that the bound given in equation (B.1) is tightest at \( d = 0 \),

\[
[\phi_{B,d}(s_0, a_g) - \phi_{B,d}(s_0, a_b)][\phi_{L,d}(s_0, a_g) + \phi_{L,d}(s_0, a_b)] \geq

[\phi_{L,d}(s_0, a_g) - \phi_{L,d}(s_0, a_b)][\phi_{B,d}(s^*, a_g) + \phi_{B,d}(s^*, a_b)].
\]

(B.2)

We now prove the following claim, that there exists a \( K > 0 \) such that, if \( K < K \), the inequality in (B.2) is satisfied.

The debt level associated with \( s^*, d^* \) is determined by the lender’s break-even condition,

\[
0 = \sum_{a \in A} p(a)(\pi(a)u_L'(y_L(a))(d^* - K) - (1 - \pi(a))u_L'(y_L(a))K),
\]

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which is
\[ d^* = K \frac{\sum_{a \in A} p(a) \pi(a) u'_{L}(y_L(a))}{\sum_{a \in A} p(a) [\pi(a) u'_{L}(y_L(a)) + (1 - \pi(a)) u'_{L}(y_L(a))]} . \]

Consequently, holding all other parameters fixed, \( d^* \) is continuous in \( K \) and
\[ \lim_{K \to 0} d^*(K) = 0. \]

It follows by the continuity of \( \phi_{B,d}(s,d,a) \) in \( d \) that if equation (B.2) is satisfied at \( d^* = 0 \) (\( s^* = s_0 \)),
\[ [\phi_{B,d}(s_0,a_d) - \phi_{B,d}(s_0,a_b)] [\phi_{L,d}(s_0,a_d) + \phi_{L,d}(s_0,a_b)] > [\phi_{L,d}(s_0,a_d) - \phi_{L,d}(s_0,a_b)] [\phi_{B,d}(s_0,a_d) + \phi_{B,d}(s_0,a_b)], \]

there exists a \( K \) such that for all \( K < K' \), the required inequality holds. This condition can be simplified to
\[ \phi_{B,d}(s_0,a_d) \phi_{L,d}(s_0,a_b) - \phi_{B,d}(s_0,a_b) \phi_{L,d}(s_0,a_d) > -\phi_{B,d}(s_0,a_d) \phi_{L,d}(s_0,a_b) + \phi_{B,d}(s_0,a_b) \phi_{L,d}(s_0,a_d), \]

which further simplifies to
\[ \frac{-\phi_{B,d}(s_0,a_d)}{-\phi_{B,d}(s_0,a_b)} < \frac{\phi_{L,d}(s_0,a_d)}{\phi_{L,d}(s_0,a_b)}. \]

Plugging in functional forms, this can be written as
\[ \frac{(y_B(a_d) + x_h)^{-\gamma_L}}{(y_B(a_b) + x_h)^{-\gamma_L}} < \frac{(y_L(a_d))^{-\gamma_L}}{(y_L(a_b))^{-\gamma_L}}, \]

or
\[ \gamma_B \ln \frac{y_B(a_d) + x_h}{y_B(a_b) + x_h} > \gamma_L \ln \frac{y_L(a_d)}{y_L(a_b)}, \]

as assumed by equation (3). We conclude that there exists a \( K' > 0 \) such that Assumption 3 holds for all \( K < K' \).

To verify Assumption 1, we require that
\[ \sum_{a \in A} p(a) [\pi(a) u_B(y_B(a) + x_h - d) + (1 - \pi(a)) u_B(y_B(a))] \geq \sum_{a \in A} p(a) u_B(y_B(a)), \]

which is always satisfied, and that the debt level \( d^* \) is less than the maximum feasible debt level, \( x_h \). By the continuity of \( d^* \) in \( K \), the latter holds for sufficiently small \( K \).

Finally, we verify Assumption 4. In the model of Section II, the set of types \( \Theta \) can be indexed by the parameter \( \rho \), and the borrower’s prior has full support on \([0, \hat{\rho}]\). Moreover, by the definition of greater weak association (equation (1)) and its equivalence to the supermodular stochastic ordering (Meyer and Strulovici (2012)), \( \theta \geq_{GW} \theta_0 \) if and only if \( \rho(\theta) \geq 0 \). It follows immediately that Assumption 4 holds.
We conclude that the model of Section II satisfies Assumptions 1, 2, 3, and 4 provided that $K < \bar{K}$ for some $\bar{K} > 0$. Hence, there exists a noncontingent equilibrium in this case, by Proposition 3.

**Proof of Lemma 1.** First note that for any values of $\theta$ for which the lenders do not offer a security, the expected utility is zero.

Proof by contradiction: Suppose that there exists a symmetric pure-strategy equilibrium such that, for some values of $\theta \in \Theta$, the security $s(\theta)$ is offered and equilibrium lender expected utility is strictly positive.

Let $\theta'$ and $s' = s(\theta')$ denote the equilibrium type and security for which lender expected utility is strictly positive. In this equilibrium, each lender earns

$$|L|^{-1} \sum_{a \in A, z \in Z} \theta'(a, z) \phi_L(s'_z, a) > 0.$$ 

Let $d'(z)$ be the function satisfying $s'_z = s_{d'(z)}$ for all $z \in Z$. Consider a deviation by some lender to the security $s''_z = s_{d''(z)}$, where $d''(z) = ad(z)$ from some $a \in (0, 1)$. By assumption, $s'' \in S$. By the monotonicity property of the borrower’s indirect utility function, $\phi_B(s_d, a)$, in $d$, we have

$$\sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s''_z, a) > \sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s'_z, a)$$

for all $\theta \in \Theta$. It follows that, regardless of the beliefs that the borrower forms off-equilibrium, she will accept security $s''$ if offered for any value of $a \in [0, 1)$.

The change in expected utility for the deviating lender is

$$\sum_{a \in A, z \in Z} \theta'(a, z)(\phi_L(s''_z, a) - |L|^{-1} \phi_L(s'_z, a)).$$

By the lower semi-continuity of $\phi_L$ in $d$ and the fact that $|L| > 1$, there exists an $a \in (0, 1)$ such that this quantity is positive. It follows that an equilibrium in which lenders earn strictly positive expected utility cannot exist.

**Proof of Proposition 2.** By Assumption 1, this equilibrium delivers sufficient utility for the borrower. Therefore, the borrower is willing to participate, and lenders earn zero profits (by the construction of $\bar{s}(\theta)$) and hence are also willing to participate.

Now consider a deviation by a single lender: suppose that a lender of type $\theta$ offers security $s'$ instead of $\bar{s}(\theta)$ and weakly profits from doing so if the security is accepted. Because the lender can weakly profit from offering this deviation, the borrower is free to place the full support of her beliefs on the lender’s true type if the security $s'$ is not offered by any type in equilibrium. If the security $s'$ is offered in equilibrium by a type other than $\theta$, then by the existence of at least three lenders, the borrower can infer the true common type. Because the security $\bar{s}(\theta)$ is on the Pareto frontier and
offers zero profit to lenders, it follows that the borrower must be weakly worse off using security \( s' \) and therefore would prefer the security \( \bar{s}(\theta) \). Because there are multiple lenders, the borrower can choose a nondeviating lender and reject the deviating security. Given that the security \( s' \) will be rejected, the lender does not profit from offering it and thus \( s(\theta) = \bar{s}(\theta) \) is an equilibrium.

**Proof of Lemma 2.** First, observe by the definition of the full-information optimal security that, for all \( \theta \in \Theta \),

\[
\sum_{a \in A, z \in Z} \theta(a, z) \phi_B(\bar{s}(\theta), a) \geq \sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s^*, a).
\]

Proof by contradiction: Suppose that the full-information optimal security is noncontingent. The full-information optimal security solves, for some value of \( \lambda > 0 \),

\[
\max_{d \in \mathcal{D}} \sum_{a \in A, z \in Z} \theta(a, z) U(d(z), a; \lambda).
\]

Suppose further that the optimal \( d'(z) = d(s^*) \) for all \( z \in Z \), with \( d^*(s) \) in the interior of \( D \). In this case, \( \lambda = \lambda^* \) and by the differentiability of \( \phi_B \) and \( \phi_L \) at that point, we must have, for all \( z \in Z \),

\[
\sum_{a \in A} \theta(a, z) U_d(d(s^*), a; \lambda^*) = 0.
\]

Because \( \theta \succeq_{GW} \theta_0 \), \( \theta \) also dominates \( \theta_0 \) in the supermodular stochastic order (Meyer and Strulovici (2012)). It follows by Meyer and Strulovici (2015) that \( \theta \) can be expressed using those authors’ “elementary transformations” \( t \in \mathcal{T} \), that is,

\[
\theta = \theta_0 + \sum_{t \in \mathcal{T}} a_t t
\]

for some constants \( a_t \geq 0 \). For all \( \theta \neq \theta_0 \), there must exist at least one \( a_t > 0 \).

Consequently, there exists some \( z' \in Z \) such that

\[
\sum_{a \in A, z \in Z} (\theta(a, z) - \theta_0(a, z)) h(a) \mathbf{1}(z \geq z') > 0
\]

for any strictly increasing function \( h(a) \), and in particular \( U_d(d(s^*), a; \lambda^*) \), contradicting the requirement that

\[
\sum_{a \in A} \theta(a, z) U_d(d(s^*), a; \lambda^*) = 0
\]

for all \( z \in Z \).

**Proof of Proposition 3.** The noncontingent security \( s^* = s_{d^*} \) has payoffs that do not depend on the index. As a result, it offers zero expected utility for the lender, regardless of the lender’s type, by the assumption that all \( \theta \in \Theta \) have the same marginal distribution with respect to the external state. By
Assumption 1, $s_{d^*}$ can deliver sufficient utility to the borrower, and thus the participation constraints are satisfied in this equilibrium. It is sufficient to rule out deviations in which a single lender offers security $s'$ instead of $s_{d^*}$, when the common type is $\theta'$, to demonstrate that this is an equilibrium.

The security $s' \in S$, $s'_z = s_{d'(z)}$, must offer strictly positive utility for the lender of type $\theta'$ (if accepted) to break the equilibrium:

$$\sum_{a \in A, z \in Z} \theta'(a, z)\phi_L(s_{d'(z)}, a) > 0.$$  

Define $\bar{\theta}$ as a type that would profit maximally from offering security $s'$:

$$\bar{\theta} \in \arg \max_{\theta \in \Theta : \mu_0(\theta) > 0} \sum_{a \in A, z \in Z} \theta(a, z)\phi_L(s_{d'(z)}, a).$$

Consider the set of “elementary transformations” $t \in \mathcal{T}$ defined by Meyer and Strulovici (2015), and suppose that for some $z', z'' \in Z$ that are adjacent in the order on $Z$ (with $z'' > z'$), $d(z'') > d(z')$. By Meyer and Strulovici (2015), we can write

$$\bar{\theta} = \theta_0 + \sum_{t \in \mathcal{T}} \alpha_t t$$

for some constants $\alpha_t \geq 0$. If there exists a $t \in \mathcal{T}$ with support on $z'$ and $z''$ such that $\alpha_t > 0$, then by the richness of the type space (Assumption 4), there exists a type $\hat{\theta} = \bar{\theta} - \beta t$, for some $\beta > 0$, such that $\hat{\theta}$ is in the support of $\mu_0$. By the submodularity of $\phi_L$ (Assumption 2),

$$\sum_{a \in A, z \in Z} \hat{\theta}(a, z)\phi_L(s_{d'(z)}, a) \geq \sum_{a \in A, z \in Z} \bar{\theta}(a, z)\phi_L(s_{d'(z)}, a).$$

Therefore, it is without loss of generality to assume that the security $d'(z)$ is weakly decreasing between adjacent pairs $z', z''$ such that $\alpha_t > 0$ for some elementary transformation with support on those pairs.

By this result and the supermodularity of the social welfare function with Pareto-weight $\lambda^*$ (Assumption 3), we must have

$$\sum_{a \in A, z \in Z} \hat{\theta}(a, z)U(s_{d'(z)}, a; \lambda^*) \leq \sum_{a \in A, z \in Z} \theta_0(a, z)U(s_{d'(z)}, a; \lambda^*).$$

By the Pareto-optimality of the noncontingent security $s^*$ under $\theta_0$,

$$\sum_{a \in A, z \in Z} \theta_0(a, z)U(s_{d'(z)}, a; \lambda^*) \leq \sum_{a \in A, z \in Z} \theta_0(a, z)U(s^*, a; \lambda^*).$$
Therefore,
\[
\sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_B(s_{d'(z)}, a) + \lambda^* \sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_L(s_{d'(z)}, a) \leq \\
\sum_{a \in A, z \in Z} \theta_0(a, z) \phi_B(s^*, a) + \lambda^* \sum_{a \in A, z \in Z} \theta_0(a, z) \phi_L(s^*, a).
\]

It follows by
\[
\sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_B(s_{d'(z)}, a) > 0 = \sum_{a \in A, z \in Z} \theta_0(a, z) \phi_L(s^*, a)
\]
that
\[
\sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_B(s_{d'(z)}, a) < \sum_{a \in A, z \in Z} \theta_0(a, z) \phi_B(s^*, a),
\]
and by the noncontingency of \(s^*\),
\[
\sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_B(s_{d'(z)}, a) < \sum_{a \in A, z \in Z} \bar{\theta}(a, z) \phi_B(s^*, a).
\]

By the \(D1\) refinement, the borrower can place the entire support of her beliefs on the type \(\bar{\theta}\). Consequently, if there exists a \(\theta'\) for which the deviation is profitable, the borrower can believe that she would be worse off and reject the deviation.

\[\square\]

\textit{Proof of Lemma 3.} By definition the set \(\Theta\) is bounded, and by assumption it is closed, and thus it is compact. It follows that \(\Theta^*(\theta)\) is nonempty. By the linearity of
\[
\sum_{a \in A, z \in Z} \theta''(a, z) \phi_L(\bar{s}_z(\theta), a)
\]
in \(\theta''\), \(\Theta^*(\theta)\) is convex.

By the assumption of ex-post efficiency, \(\bar{s}_z(\theta) = s_{d(z, \theta)}\) for some \(d(z, \theta) \in D\). By definition,
\[
d(z, \theta) = \arg \max_{d(z) \in D} \sum_{a \in A, z \in Z} \theta(a, z) \phi_B(s_{d(z)}, a)
\]
subject to
\[
\sum_{a \in A, z \in Z} \theta(a, z) \phi_L(s_{d(z)}, a) \geq 0.
\]

By assumption, \(\phi_L(s_d, a)\) and \(\phi_B(s_d, a)\) are continuous in \(d\), and hence it follows that \(d(z, \theta)\) is continuous in \(\theta\) and that
\[
\sum_{a \in A, z \in Z} \theta''(a, z) \phi_L(\bar{s}_z(\theta), a)
\]
is jointly continuous in \((\theta, \theta'')\). Therefore, by Berge’s theorem (the theorem of the maximum), \(\Theta^*(\theta)\) is upper semi-continuous.
It follows that Kakutani’s fixed point theorem holds, and hence there exists a \( \theta^* \) such that

\[
\theta^* \in \Theta^*(\theta^*),
\]
as claimed.

Now suppose that there is another \( \theta' \in \Theta^*(\theta^*) \). We must have

\[
\sum_{a \in A, z \in Z} \theta'(a, z)\phi_L(s_{d(z, \theta)}', a) = 0.
\]

Because the index and the external state are perfectly correlated, we can rewrite this expression as

\[
\sum_{a \in A, z \in Z} q(z; \theta') \delta(a, z)\phi_L(s_{d(z, \theta)}', a) = 0,
\]

where \( q(z; \theta') \) is the marginal distribution associated with \( \theta' \).

By the definition of \( \bar{s}(\theta) \), for all \( z \in Z \) (by the full support assumption, \( q(z; \theta^*) > 0 \)),

\[
\phi_B(s_{d(z, \theta')}, a) + \lambda \phi_B(s_{d(z, \theta')}, a) \geq \phi_B(s_{d}, a) + \lambda \phi_B(s_{d}, a)
\]

for all \( d' \in D \) and some multiplier \( \lambda > 0 \). It follows that optimality would also hold if \( d(z; \theta') = d(z; \theta^*) \), and feasibility is satisfied, and therefore

\[
\bar{s}(\theta') = \bar{s}(\theta^*),
\]
as required. \( \square \)